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Enriching math teaching guides from a competency-based perspective

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Abstract
This study introduces a novel framework designed to enrich mathematics teaching guides from a competency-based perspective. First, we narrow down the concept of a teaching guide in mathematics education, grounded in the documentational approach to didactics. This definition offers an updated perspective on the structure and function of math teaching guides in educational settings. Second, we provide a comprehensive definition of 'richness' in math activities, encompassing content, processes, cognitive demand, and classroom management. Lastly, we introduce an analytical tool developed for assessing and enhancing the richness of math teaching guides. This tool, formed through theoretical analysis and empirical testing, assists educators and curriculum developers in creating more balanced and integrative teaching guides. The results suggest that the tool holds potential for broader applications in curriculum design and teacher education. The findings contribute to the broader understanding of how teaching guides can effectively capture and communicate the richness of activities, thereby serving as a valuable tool for improving mathematical education resources.

Keywords: mathematics education, teaching guides, competency-based education, curriculum enrichment, instructional design, documentational approach to didactics

INTRODUCTION

The role of resources, documents and curriculum design in education has been a subject of extensive research, with researchers acknowledging their importance in shaping learning experiences for both students and teachers (Remillard, 2005; Trouche et al., 2018). Particularly, *teaching guides* (as defined in our theoretical framework), used by teachers to facilitate activities, serve as a critical link between curriculum design and classroom implementation. In fact, teaching guides significantly influence teacher performance, serving as both an operational tool for implementing activities and providing classroom management advice, and as a resource for continuous professional development. Their pivotal role extends to curriculum designers as well, constituting a primary means of communication with teachers, who interact with these guides on a weekly basis. Different authors have explored various aspects of textbooks (Gueudet et al., 2013) and teachers' use of math curricula (Remillard, 2005), and even studied how teacher guides give support to teachers (Jukić Matić & Glasnović Gracin, 2021). Nevertheless, there remains a lack of a comprehensive and specific definition of what constitutes a *teaching guide* (opposed to a textbook or any other curriculum resource). Such absence leaves room for varied interpretations and can lead to inconsistencies in research and teaching practices. This study starts by proposing a concrete definition for *teaching guides*, grounded in the documentational approach to didactics (DAD) framework (Trouche et al., 2018).

The term *rich math task* is broadly used by math teachers and experts (e.g., Math For Love, 2022), curriculum designers (e.g., Gojak, 2017), governments (e.g., Virginia Department of Education, 2023), and prestigious math education institutions (e.g., Piggott, 2007). While analyzing math tasks and its quality is a vast field of research (Margolinas, 2013), the definition of richness, or enrichment, remains nebulous. Specifying such definition with a comprehensive approach that combines the main aspects of enrichment (both task and management) to analyze and improve teaching guides could be innovative.

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Contribution to the literature

- This study offers a novel definition of a teaching guide as a document, grounded in DAD framework (Trouche et al., 2018), providing a fresh perspective on how guides can be conceptualized and studied.
- It identifies key traits of enrichment, drawing us closer to a comprehensive definition of richness in the context of math teaching guides. These traits encompass content, processes, cognitive demand, and management, among others, offering a multifaceted view of what constitutes a rich math guide.
- It presents the development and validation of a framework that facilitates the analysis and enrichment of math teaching guides. This tool, with its robust theoretical underpinnings and practical utility, enhances our ability to evaluate and improve the richness of math teaching guides.

Our research aims to fill these gaps by proposing a novel framework for analyzing and enriching teaching guides from a competency-based perspective (combing ideas by de Lange's 1999 works on cognitive demand, Niss & Højgaard, 2019, and Schoenfeld's TRU-math, 2016). This framework integrates key dimensions of enrichment, including content, processes, and class management, offering a more holistic perspective on guide evaluation and improvement. By doing so, we hope to contribute to the ongoing discourse on math guides design, providing educators and curriculum designers with a solid tool to enhance the richness of their guides.

To illustrate our approach, we draw upon a metaphor likening activities to reality and guides to films. Just as a film captures a fraction of the richness and complexity of reality, a guide encapsulates a portion of the richness inherent in an activity. This metaphor not only underscores the complexity of activities and the relative simplicity of guides, but also highlights the potential of our framework to capture and communicate the richness of activities when written in a guide format (i.e., as a document).

In this context, our research question is: How can we develop and apply a comprehensive framework that integrates key dimensions of math enrichment to effectively analyze and improve teaching guides?

In the following sections, we will delve deeper into our framework, detailing its development, application, and implications for guide enrichment. We will also present a case study, where we apply our framework to a guide from Innovamat's curriculum (Vilalta, 2021), providing a practical demonstration of its utility, potential impact and limitations.

THEORETICAL FRAMEWORK

This study starts from DAD by Trouche et al. (2018), exploring the complex relationship between teachers and the resources they use in their teaching practice. This understanding forms the foundation of our approach. Within this context, we specifically focus on defining and analyzing math teaching guides, which are distinct from traditional textbooks. We consider the intrinsic quality of these guides, their coherence, the tasks and the class management they propose.

Secondly, we explore and define the concept of rich math activities from a competency-based perspective, drawing upon Niss and Højgaard's (2019) definition of mathematical competence, the characterization of a powerful class by Schoenfeld's (2016) TRU-math and de Lange (1999) works on cognitive demand.

This framework guides our development of an analytical tool designed to characterize the richness of teaching guides, which will be discussed in the following sections.

Meaning of *Document*

The term *resource* in educational environments can refer to a wide variety of tools and formats. DAD is a theoretical framework that allows for the analysis of the complex relationship between teachers and these resources. According to Trouche et al. (2018), a mathematics curricular resource is 'any resource (e.g., digital interactive, nondigital/traditional text) that are developed and used by teachers and pupils in their interaction with mathematics in/for teaching and learning, inside and outside the classroom' (p. 3).

They further elaborate by distinguishing between *material curriculum resources* (e.g., textbooks, digital resources, manipulatives, and calculators), *social resources* (e.g., a conversation on social networks or in a forum), and *cognitive resources* (e.g., any theoretical framework used to work with teachers).

Teachers interact with these resources in what DAD calls the *scheme of use*, which is different for each teacher, depending on their experience and knowledge. A scheme has four components:

(a) the objective of the teaching activity,

(b) the rules of action, information gathering, and classroom control,

(c) operational invariants (propositions considered as true and other relevant concepts), and

(d) possibilities for inference and adaptation to various situations.

Thus, the combination of the resource and its scheme of use generates the *document*, the cornerstone of DAD.

Table 1. Key differences between a textbook & a teaching guide

Point of view		Textbook	Teaching guide
1	DAD	Resource	Document (resource with a scheme of use)
2	Voice	Through teachers	To teachers
3	Focus	Content & tasks	Tasks, questions and management (i.e., activities)

The interaction between a teacher and a resource is bidirectional: the resource influences the teaching practice (*instrumentation process*), while the teacher's knowledge conditions the use of the resource (*instrumentalization process*). As an example, we can imagine a teacher using a traditional textbook. Units and proposed tasks deeply shape how this teacher is teaching math in the class. At the same time, teacher's experience, knowledge and even classroom conditions also transform how the textbook is used, whether tasks are literally followed or not, whether learning goals are modified or not, etc. This interaction results in a dynamic, evolving process known as *documentational genesis*. As Gueudet and Trouche (2009) describe,

> A document developed from a set of resources provides new resources, which can be involved in a new set of resources, which will lead to a new document etc. Because of this process, we speak of the dialectical relationship between resources and documents (p. 206)

Defining & Analyzing a Teaching Guide

This study aims to characterize *richness* from a competency-based perspective in order to develop an analytical tool that allows for the observation of teaching guides from this viewpoint. Such a goal leads us to the prior need of defining what traits constitute a teaching guide, as opposed to other resources, especially textbooks.

Usually, DAD understands textbooks as resources, not documents, because they typically lack a scheme of use. In contrast, a *teaching guide* (hereafter, *guide*) is not a resource but rather a *document*. We can affirm that because, apart from the tasks for each session, a guide also details the objectives of the session, rules of action, and some adaptations, key traits of a scheme of use (Gueudet & Trouche, 2009).

As Remillard (2005, 2012, 2013) has explored, there are resources that talk *through* the teacher (i.e., to the students) and others that talk *to* the teacher. This characteristic is what the author calls the 'voice' of the resource. Here, we find a second key difference between a textbook and a guide: while a textbook talks through the teacher aiming the students, a guide talks to teachers.

This difference in the voice has implications in the focus. A textbook typically explains content theory with texts, diagrams, and drawings, as well as proposing tasks to work on this theory. On the contrary, a guide is more focused on describing the different tasks and questions to be proposed to students, often with a didactic justification, rather than just explaining the theory. Additionally, it includes classroom management tips, ranging from grouping proposals to recommendations for addressing both struggling students and talented ones.

Table 1 summarizes the three explored differences between a textbook and a guide.

There are other aspects of curricular materials that could be explored, like their visual aspect (what Remillard, 2005, calls 'look'), their structure, or other communicative aspects. Nevertheless, we find them not as relevant for the purpose of defining a guide and exploring its richness from a competency-based perspective.

As Jukić Matić and Glasnović Gracin (2021) claim, 'compared to the research on textbooks, research on teacher guides is sparse' (p. 1). Moreover, most authors (e.g., Ahl et al., 2015; Brown, 2009; Jukić Matić & Glasnović Gracin, 2021) focus on the teacher-guide interaction (i.e., when the teacher is using the guide), rather than analyzing the richness of the guides itself. In any case, although the differences between what is understood as a textbook and what is considered a guide are evident, they also present certain similarities that make much of the research done around textbooks applicable.

The first thing to keep in mind, as also argued by Shield and Dole (2012), is that the analysis of a textbook (like the analysis of a guide) can only inform about its potential to generate learning, since its use depends on the teacher. Therefore, some authors, such as Trouche et al. (2013), begin their analyses by explaining that the *intrinsic quality* of a resource must be distinguished from its adequacy and subsequent use. This intrinsic quality encompasses mathematical, didactic, and user experience aspects.

Gueudet et al. (2013) also discuss the *conceptual coherence* of the textbook, which can be applied to a guide too. According to the authors, this can be understood as a combination of the correctness of mathematical content, alignment with the official curriculum, appropriate sequencing of concepts, and the correct relationship between the content covered and the tasks proposed.

Other models for analyzing traditional textbooks (Valverde et al., 2002) are based on three aspects: content (number, geometry, etc.); performance expectations (using routine procedures, solving problems, etc.); and perspective (attitudes, participation, interest, habits, etc.). As Gueudet et al. (2013) explain, though, most of these analyses focus on tasks and problem-solving procedures (Fan & Zhu, 2007; Zhu & Fan, 2006), procedural complexity (Vincent & Stacey, 2008), cognitive demand (Jones & Tarr, 2007) and depth of understanding in terms of making connections, as described by Pepin (2008). It is difficult to find analysis that explore richness from a *complete* competency-based perspective (as defined in the following section), mainly because textbooks tend to consider only content, rather than math processes, competencies and class management. In fact, problem-solving, connections or cognitive demand are, as we will see below, aspects to consider when talking about competency-based richness, but others that we consider relevant are missing, such as processes like 'reasoning and proof' or 'communication and representation' (as described by National Council of Teachers of Mathematics [NCTM], 2000) and, above all, classroom management considerations (one of the key differences between a textbook and a guide).

We propose, therefore, to bring together several of these ideas in a framework that allows for the characterization of the potential competency-based richness (i.e., intrinsic quality) of a document (a resource with a schema of use) such as a guide formed by activities.

Rich Math Activities from a Competency-based Perspective

The expression *rich math task* is a phrase commonly employed across various spheres of the educational sector. Teachers and experts in the field, such as Finkel (Math For Love, 2022), have embraced this terminology: 'Every student deserves to have the opportunity to problem-solve and engage in genuine mathematical thinking. *Rich tasks* are designed to make these rich learning experiences possible.' Similarly, curriculum developers such as Gojak (2017) also use the term: 'A rich task presents a high level of cognitive demand and requires students to think abstractly in order to make connections to and among mathematical concepts' (p. 3). Governmental bodies like the Virginia Department of Education (2023) describe it too: 'Rich mathematical tasks engage students in sense-making through deeper learning that require high levels of thinking, reasoning, and problem solving'. Furthermore, esteemed academic institutions like Cambridge University (Piggott, 2007, 2011), also reference this term. Despite its widespread use, the precise meaning of what constitutes 'richness' in mathematics education is still somewhat elusive. While analyzing math tasks and its quality is a vast field of research (Margolinas, 2013), 'richness' remains rather unexplored from a research point of view.

Before diving into that, though, we need to agree on what a *task* is. According to the authors of the ICMI 22 monograph on task design, edited by Margolinas (2013), the concept of *task* has been used with different meanings over the years. For Leont'ev (1975), one of the pioneers in the use of the term, a task is any operation performed under specific conditions. The same authors of the monograph, however, use the term in a very broad way:

> We use task to mean a wider range of 'things to do' than this, and include repetitive exercises, constructing objects, exemplifying definitions, solving single-stage and multi-stage problems, deciding between two possibilities, or carrying out an experiment or investigation. Indeed, a task is anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something. Tasks can also be anything that students decide to do for themselves in a particular situation. Tasks, therefore, are the mediating tools for teaching and learning mathematics and the central issues are how tasks relate to learning, and how tasks are used pedagogically (Margolinas, 2013, p. 9-10).

Other authors, such as Christiansen and Walther (1986) or Mason and Johnston-Wilder (2006), advocate for a more limited definition, similar to the one used by Vilalta et al. (2021), defining *task* as what is asked to the students, while *activity* refers to the interaction between students, teachers, resources, and the environment surrounding the task. In our work, we adopt this more concrete definition: a task is the premise, the statement, the demand made to students. A task, or a set of tasks, combined with the management by the teacher as they interact with the students, is what we understand as *activity*, which can be richer or less.

$$\textit{Activity} = \textit{Tasks (demands for students)} + \textit{management (interactions)}. \quad (1)$$

When considering this equation, it is hard not to establish parallels between a *document* (understood as the sum of a resource and a usage scheme) and an *activity* (the sum of one or more tasks and their management): we could argue that any task can be understood as a specific type of resource, and that its management is precisely the scheme of how this resource is used. This parallelism is especially relevant for our work, because it strengthens the conception of guides (formed by activities) as documents.

Once we have agreed that *task* and *management* are the two ingredients of an *activity* (and therefore, of a guide made of activities), it is appropriate to describe traits that characterize its richness. The aforementioned author, Piggott (2007, 2011), within the framework of the NRICH project at Cambridge, proposes a practical framework to approach richness. It includes two factors: *content* and *teaching*. According to Piggott, rich activities require, on the one hand, content based on appealing problems that foster the development and use of strategies and mathematical thinking; and, on the other hand, a teaching approach (i.e., classroom management) that encourages an open and flexible environment in which cooperative work, exploration, and communication are promoted and where difference is used as a learning tool. The two factors described by Piggott have a clear relationship with the two elements of a rich mathematical activity described by Vilalta et al. (2021) based on Deulofeu and Vila (2021): the task and its management. This is why the analysis of richness makes much more sense when talking about teaching guides because, unlike textbooks, these include classroom management recommendations.

To analyze this 'development and use of strategies and mathematical thinking' referred by Piggott, we propose to start with what the literature says about 'learning mathematics with understanding' (Hart and Team, 1981). According to Hiebert et al. (1997), students construct understanding by 'reflecting and communicating', so rich tasks must promote these *processes*:

> First, the tasks must allow the students to treat the situations as problematic, as something they need to think about rather than as a prescription they need to follow. Second, what is problematic about the task should be the mathematics rather than other aspects of the situation. Finally, in order for students to work seriously on the task, it must offer students the chance to use skills and knowledge they already possess. Tasks that fit these criteria are tasks that can leave behind something of mathematical value for students (p. 8).

What in the eighties was called *learning with understanding* could now be understood as *developing mathematical competence*. Thus, following the ideas of the PISA theoretical framework (Niss & Højgaard, 2019; Organization for Economic Co-operation and Development [OECD], 2017a) and the *process standards* defined by NCTM (2000), we understand that mathematical competence consists of four processes: problem-solving; reasoning and proof; connections; communication and representation. These processes, widely discussed and accepted by the math education community, structure mathematical work beyond content and give it meaning. Although each territory interprets them differently in their official curricula, leading to small variations, different countries such as the United States (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010), Australia (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2015), Hong Kong (The Curriculum Development Council of Honk Kong, 2017) or Spain (LOMLOE, 2020), among others (Thompson et al., 2018) are already adopting a competency-based conception of mathematics. It is not the objective of this article to delve into the causes of this conception or to discuss it. In any case, thanks to the processes of mathematical competence, we can characterize a first indicator of richness: the task must provide opportunities to develop one or more of these mathematical processes. It is hard not to observe parallels between this process-based conception of math with some of the aspects to analyze a textbook exposed in the previous section: for instance, Valverde et al. (2002) talk about performance expectations, and they distinguish using routine procedures from *solving problems*; Fan and Zhu (2007) and Zhu and Fan (2006) focus on tasks and *problem-solving procedures*; Pepin (2008) conceives understanding in terms of making *connections*.

At the same time that we highlight the importance of working on processes, we cannot ignore the content. A second indicator of richness is found, as described by Deulofeu and Vila (2021) within the mathematical content involved in the task. According to the authors, this must be *contextualized* (relevant to the student), *rigorous* (correct, with mathematical sense and in accordance with the curriculum), and *extensible* (connected to prior and future knowledge). Again, all three ideas are directly related to Gueudet et al.'s (2013) 'conceptual coherence' when analyzing a textbook: correctness, alignment with the curriculum, and sequence. Valverde et al. (2002) also consider content as a key aspect to analyze in a textbook.

Other authors do not literally speak of richness, but they also describe indicators to enrich mathematical activity in the classroom. According to TRU-math (teaching for robust understanding) by Schoenfeld (2016), the *five dimensions of a powerful class* are: the *content*; the *cognitive demand*; the *equitable access to content*; the *agency, ownership and identity*; and the *formative assessment*. Although these dimensions refer to a class, many of the ideas presented by Schoenfeld (2016) are very useful when it comes to complementing the analysis of the richness of a document, such as a guide, both in terms of the task (resource) and its management (scheme of use).

Without going any further, the first dimension speaks of the *content*, which must 'provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers' (p. 4). We can see that the author is including competency-based ideas (i.e., 'resourceful disciplinary thinkers') into his understanding of content. Since the conceptualization of the present approach to richness comes from a competency-based perspective, for analytical purposes we prefer to separate the processes (problem-solving; reasoning and proof; connections; communication and representation) from the specific math content blocks (numbers and operations; space and shape; measurement; statistics and probability; change and

Table 2. Components of a rich math activity from a competency-based perspective

Richness components of an activity		Description
1	Processes	The task promotes one or more of the four processes: problem-solving, reasoning and proof, connections, communication and representation.
2	Contents	The task fosters content that is contextualized, rigorous and extensible.
3	Cognitive demand	Task requires reflecting rather than relating & relating rather than reproducing.
4	Differentiated instruction	The management considers access and challenge for all students.
5	Environment	The management fosters engagement and autonomy, develops math identity, promotes peer learning, etc.

relationships). This specific naming for the blocks comes from Innovamat's framework (Vilalta, 2021), which originally follows the adaptation from the US common core (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010) made by the Catalan curriculum (Burgués & Sarramona, 2013). Nevertheless, once it is understood that 'space and shape' includes geometry, localization and orientation, while 'change and relationships' involves algebraic thinking such as identifying patterns and describing regularities, these blocks are very similar to any other math curriculum.

The second dimension by Schoenfeld (2016), *cognitive demand*, is directly observable from the task. By cognitive demand, we understand the level of challenge posed to students. This is close to Gueudet et al.'s (2013) *depth of understanding*; Vincent and Stacey's (2008) *procedural complexity*; and obviously Jones and Tarr's (2007) *cognitive demand.* According to Schoenfeld (2016), 'students learn best when challenged in ways that provide space and support for growth, with difficulties for tasks ranging from moderate to demanding' (p. 5). To refine the cognitive demand levels of a task, we turn to the work of de Lange (1999) in adapting the national Dutch option of TIMSS (Boertien & de Lange, 1994; Kuiper et al., 1997): level 1, the most basic, refers to purely reproductive tasks, repetition, such as applying a standard algorithm or formula or evoking a definition; level 2 speaks of tasks that require establishing connections between different domains of mathematics and integrating information; level 3, the highest, refers to tasks, where it is necessary to mathematize a situation, reflect, develop new strategies, etc. To facilitate future references, we will refer to the three levels with three verbs starting with 'r': 'reproduce', 'relate', and 'reflect', respectively. Thus, for a task to be considered rich, it must pose an appropriate cognitive demand, more reflective than reproductive.

The remaining three dimensions by Schoenfeld (2016), on the other hand, are more related to management than to the task itself.

The third dimension, *equitable access to content*, highlights the importance of promoting the participation of all students: 'Classrooms in which a few students get most of the *airtime* are not equitable, no matter how rich the content: all students need to be involved in meaningful ways' (p. 7). Authors such as Bartell et al. (2017) are very forceful in this aspect and argue that teaching practices without explicit attention to equity are 'inevitably doomed to failure'. In fact, from Piggott (2011) we already know that using difference as a learning tool is key to enrich math activities.

The fourth dimension, *agency, ownership,* and *identity*, refers to the opportunities we offer students to take ownership of the task, engage in it, and develop their identity as thinkers and learners, both individually and collectively. Valverde et al. (2002), when defining aspects to analyze a textbook, also refer to what they call *perspective*, a combination of the attitudes, participation, interest, habits, etc. promoted by the book. We call this the *environment* set by the activity.

Finally, the fifth dimension is *formative assessment*. For Schoenfeld (2016), a formative assessment is one that 'meets students, where they are and gives them opportunities to deepen their understandings' (p. 11). Since assessment is a delicate field of research that would deserve exclusive treatment, however, for the purposes of this study, we will not consider it in our richness traits.

Although, as we have said, some dimensions by Schoenfeld (2016) refer to the task and others to the management of the activity, the two aspects often mix, making the analysis of activities difficult. For example, a specific management approach can alter the cognitive demand at play, which initially depends on the task. Or a task may have a statement that makes it more accessible, when accessibility is an aspect that falls more on the management side.

In any case, according to the aforementioned framework, we consider that richness is defined by five main components (**Table 2**). Obviously, as discussed in the limitations section, there are many other aspects that could be considered as richness traits. One first example, as said before, would be the assessment of the activity. Does the activity focus on assessing only contents, or it assesses also processes? Does it allow teachers to assess from a formative perspective? Another example: recently, some authors and institutions (OECD, 2017b) have discussed the socioemotional skills that come into play when doing mathematics in the classroom, from transversal skills such as cooperation to math error management or resilience to frustration.

However, this is still an emerging field of study, which goes beyond the main objectives of this article. We believe, in any case, that it would be highly relevant to add such aspects to the analysis and characterization of richness in future iterations of our components and therefore, our *definition of richness* from a competency-based perspective.

METHODOLOGY

Context

This study is part of a Spanish industrial doctorate program, a model that fosters close collaboration between academia and industry. In this model, our research is conducted within a company setting, with the aim of addressing real-world, industry-relevant problems. This approach combines the theoretical rigor of traditional academic research with the practical, problem-solving orientation of industry work. This model is similar to the engineering doctorate (EngD) in the UK or industry-oriented doctorates in other European countries. The company hosting the PhD student, relevant to the context, is Innovamat Education. As described in Vilalta (2021), Innovamat is a project that provides evidence-based resources to teach competency-based mathematics in both elementary and high school. Since its inception, the industrial doctorate has been perceived as an opportunity to carry out research within the company, to contribute to the teaching of mathematics as a science and, ultimately, to detect opportunities to improve the Innovamat's resources offered to school.

One of the cornerstones of this curriculum provider are its *teaching guides*, a written document that, with the intention of being formative for teachers (as considered by Leikin & Zazkis, 2010), sets out the sequence of tasks for each session and the recommended management in the classroom. The present study aims to develop an analysis tool that allows characterizing any guide from the point of view of intrinsic competence richness. As discussed in the theoretical framework, the guides are made up of activities in what can be considered a document in the sense that DAD gives them: it is a resource (the set of tasks proposed for each session) accompanied by a scheme of use (its management). It should be mentioned that, while DAD suggests that the scheme of use is provided by the teacher when using the resource, we argue that in the activities of the guides, much of this scheme (i.e., the objective, rules of action, and some adaptations) is explicitly provided, making them more of a document than a resource. However, this does not prevent each teacher from interpreting and using the guides differently.

The relationship between activities and guides can be likened to the relationship between reality and a film. Activities, composed of tasks and their management, exist in the dynamic real world. Guides, on the other hand, are akin to films, simple representations of this reality. Just as a film can only capture a fraction of the richness and complexity of reality, a guide can only encapsulate a portion of the richness inherent in an activity. Diving into the metaphor, we could say that the content is akin to the brightness, illuminating the key elements and making them clearly visible. The processes add color, bringing depth and nuance to the scene. Management, then, is like the movement, providing the dynamic interaction that brings the scene to life.

Typically, a textbook only includes contents so, going back to our metaphor, it would be like a black and white picture, capturing the basic elements but lacking in color and dynamism. A guide that includes both contents and some management would be like a black and white film, adding movement and interaction but still without color. In contrast, a guide that includes contents and processes, but not management, is like a color picture, offering a richer view but still static. Finally, a guide that includes task content, processes, and management would be like a color film, providing a more comprehensive and dynamic representation of the activity.

The richness of the activity is a necessary condition for the richness of the guide, much like an interesting reality is necessary for an interesting film. Nevertheless, this is not a sufficient condition. A rich activity does not automatically translate into a rich guide. The quality of the guide, like the quality of a film, depends on how well it captures and communicates the richness of the activity. For experienced teachers, who are familiar with the activity, a less detailed guide might suffice, as they can fill in the gaps with their knowledge and experience. They can infer the richness of the activity from a sparse guide, much like someone familiar with a landscape can infer full view from a limited film. But, for less experienced teachers, a rich guide is crucial to fully understand and implement activity effectively, just as a detailed film is necessary for someone unfamiliar with landscape to appreciate its full beauty and complexity.

Of course, like most metaphors, this one could be extended: Would the social emotional skills be akin to the sound of a film, because they add a layer of emotional resonance, enhancing the overall experience and making the scene more engaging and relatable? However, such extensions are unnecessary for the purposes of this study. Instead, we focus on the value of this metaphor in its ability to highlight the complexity of activities and the relative simplicity of guides. Activities, akin to reality, are multifaceted and complex, making them challenging to characterize comprehensively, especially when considering both task and management aspects. Guides, however, are composed of discrete segments, much like the pixels in an image or video, which makes them more manageable to analyze. Such approach contributes to the broader theoretical understanding of how guides can effectively capture and communicate the richness of activities.

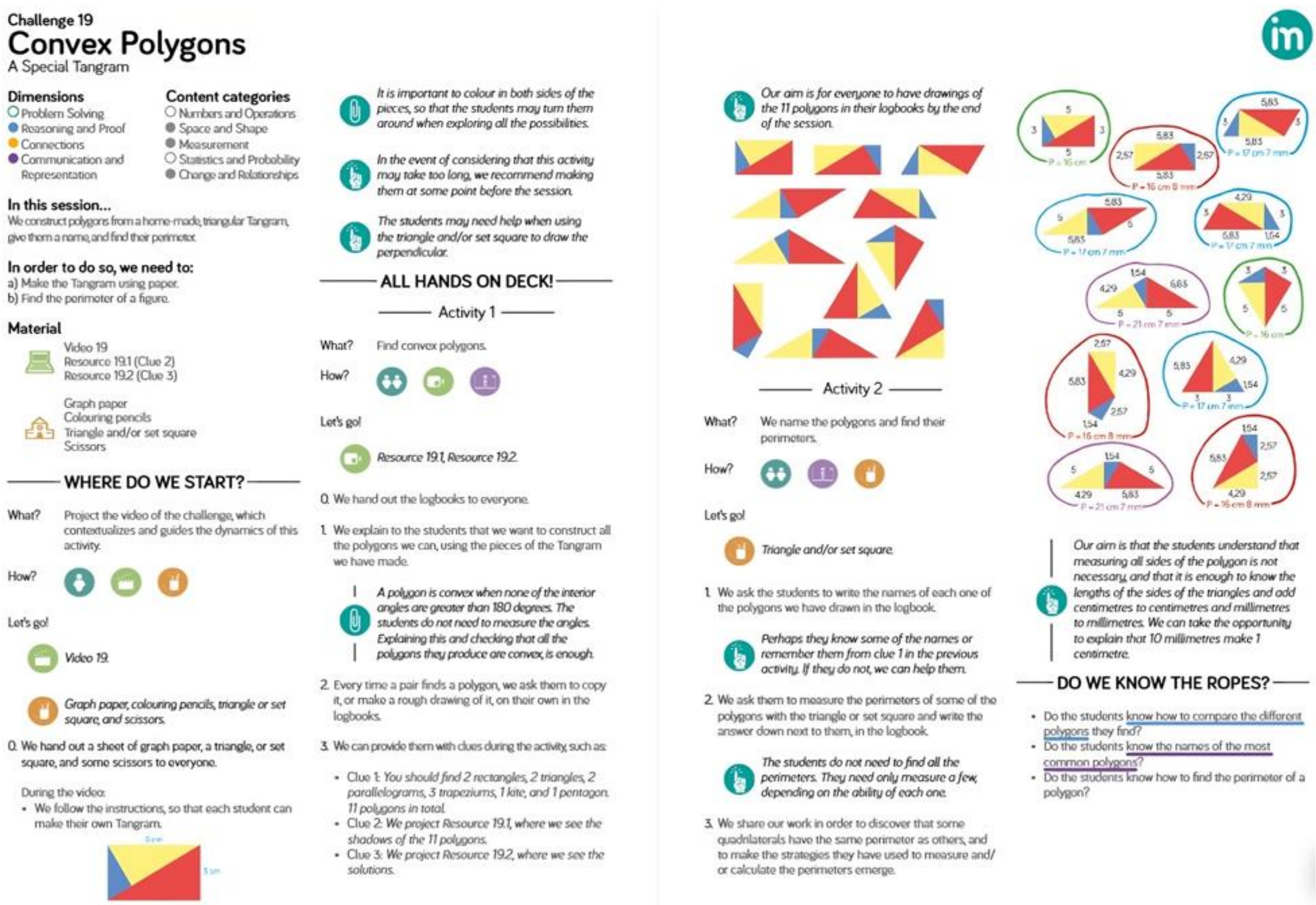


Figure 1. Innovamat's teaching guide for a 4th grade session (Innovamat, 2021)

An Innovamat teaching guide is a document that provides a sequence of activities for each session (i.e., tasks along with recommended classroom management strategies). As an example, here we can see a specific session for 4th grade (**Figure 1**).

In this math session, students are introduced to the concept of convex polygons and perimeter measurement through a hands-on activity involving a geometric puzzle consisting of three pieces. The session commences with the students crafting their own triangular Tangram by following a projected video guide. Once the tangrams are prepared, the students work in pairs to construct various convex polygons using the pieces. Guidance and prompts are provided to help them discover a range of shapes including rectangles, triangles, parallelograms, trapeziums, a kite, and a pentagon.

As the students identify each polygon, they are encouraged to sketch these shapes in their logbooks for future reference. Clues are given throughout this process to aid in the discovery and identification of the shapes. In the second part of the session, students label their identified polygons and learn how to measure their perimeters using a triangle or set square. This exercise is not only about obtaining the measurements, but also designed to help students discover that some polygons do share the same perimeter while others do not. By the end of this session, the aim is for students to have developed an understanding of convex polygons and their properties, as well as measuring perimeters.

Data Collection: Defining the Tool

Our analytical tool was developed based on the theoretical framework, to characterize the richness of a teaching guide from a competency-based perspective. Our definition of richness refers to activities (understood as task plus management). Since we understand that a guide is formed by activities, we consider that the richness of the guide is directly derived from the richness of the activities that make it up.

The characterization starts by considering the minimum unit of information with meaning within the guide, what we call a *segment* of the guide (this can be a sentence, a step, a picture, etc.). First, we focus on the *type* to distinguish between segments that are *instructions*, those that require some actions from the teacher, and segments that are *information*, and therefore they do not ask for any specific actions. From there, both instructional and informational segments can refer mainly to *task* or to its *management*. The following diagram gathers different aspects of richness (**Figure 2**).

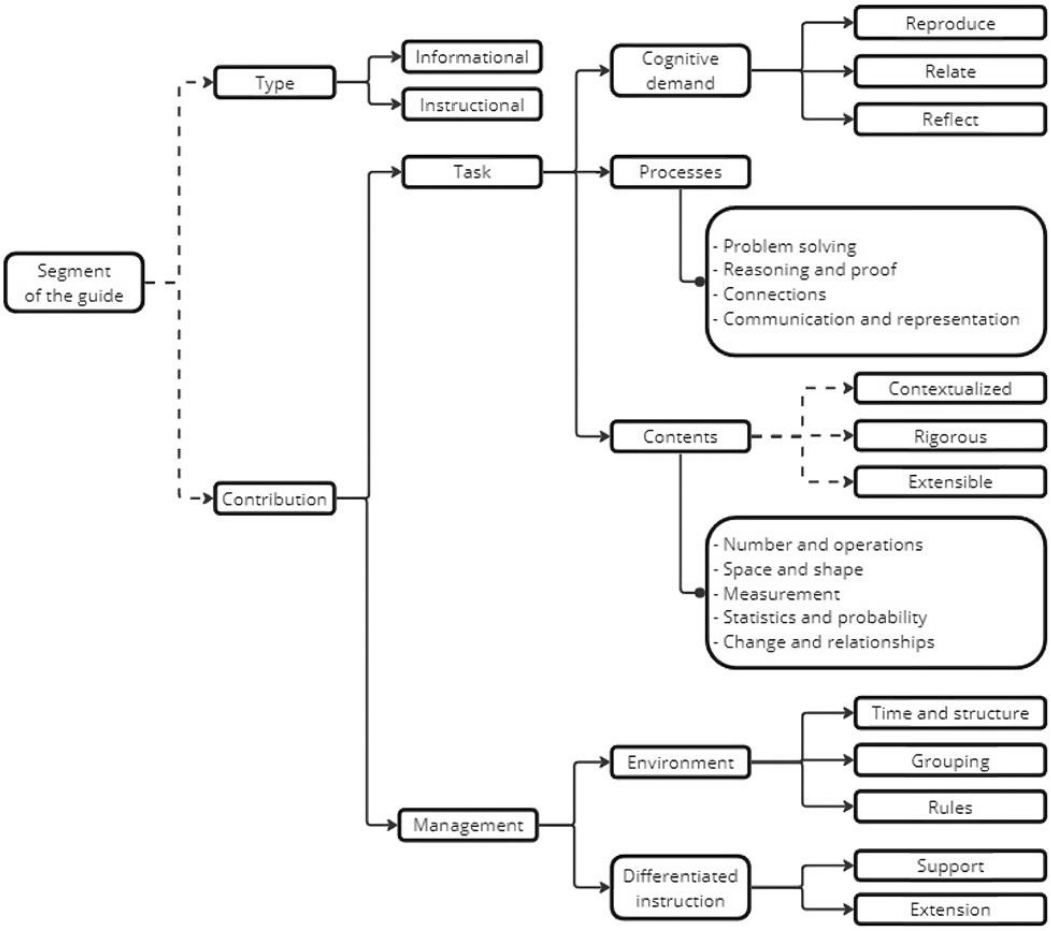


Figure 2. Diagram with aspects to characterize a segment of a guide (Source: Authors' own elaboration)

Now that we have a big perspective on which are the aspects to consider, we can dive into how we understand them. Most of the aspects have exclusive characteristics, while others do not (dashed links in the previous diagram). For instance, a segment's type is either informational *or* instructional, a segment may contribute more to the task *or* to its management, etc. On the contrary, other aspects such as the segment's content should be characterized as contextualized, rigorous *and* extensible at the same time.

Table 3 might be useful to understand those differences. Once the tool was developed, testing it was necessary in order to detect inconsistencies and improve it. At this early stage, we decided to test the tool by applying it to a specific session from the Innovamat 4th grade curriculum, known as *challenge 19* (its guide was introduced before, see **Figure 1**).

Why this session? Firstly, the article main author's recent teaching experience was focused on 3rd and 4th grade, so such grades were prioritized to facilitate the analysis. Secondly, among the different types of session included in the program, as described in Vilalta (2021), *adventures* like *challenge 19* are independent of previous and following sessions. This makes its analysis less complex. Thirdly, Innovamat's authors exposed that *challenge 19*, inspired by a relatively known activity on Brügner's (1984) minimum tangram, was a representative example from their program.

In the analysis section, we deepen into how the tool was applied by three different experts in mathematics education to validate it or find discrepancies and improve it. In any case, the tool should work regardless of the session chosen, so further iterations with different guides and programs would help in refining the tool, as discussed in the limitations section.

DATA ANALYSIS & RESULTS

In this section, we delve into application of our tool. It was applied to a specific guide (*challenge 19* from Innovamat's 4th grade curriculum) by three different experts in mathematics education. This process of *triangulation* (Flick, 2013), where different experts independently apply the tool to the same guide, allows us to preliminarily analyze the consistency and reliability of our tool. This analysis will shed light on the potential of the guide to facilitate rich mathematical activities with an effective classroom management and will provide insights into practical application of our tool.

In order to apply the tool to a guide, we start by atomizing the session guide into segments, the minimum amount of information with global meaning.

Table 3. Segment's aspects & their definition of richness

AS	Options	Richness characterization	Main source(s)
Type	Informational Instructional	Guides should combine both types of segments: Instructional ones summarize main steps, while informational ones complement them.	-
Contribution	Task Management	There should be a balance between segments contributing to task & to its management.	Activity definition by Vilalta et al. (2021) & task definition by Mason and Johnston-Wilder (2006)
Task: Cognitive demand	Reproduce Relate Reflect	Guides should involve high cognitive demand tasks (relate & reflect over reproduce).	Schoenfeld's (2016) second dimension & cognitive demand levels by de Lange (1999)
Task: Processes	• Problem-solving • Reasoning & proof • Connections • Communication & representation	Guides should involve tasks that foster one or more of these processes.	Competencies by Niss and Højgaard (2019), processes standards by NCTM (2000), & performance expectations by Valverde et al. (2002)
Task: Content	Contextualized Rigorous Extensible • Numbers & operations • Space & shape • Measurement • Statistics & probability • Change & relationships	Guides should involve contextualized, rigorous, *and* extensible content from curricular content blocks.	Content characteristics by Deulofeu and Vila (2021), coherence by Gueudet et al. (2013), content blocks by National Governors Association Center for Best Practices, & Council of Chief State School Officers (2010), & Schoenfeld's (2016) first dimension
Management: Environment	Time & structure	Guides should be structured, including recommendations on how to sequence tasks, use of manipulatives, & time distribution along session.	Sequencing by Gueudet et al. (2013)
	Grouping	Guides should advise on how to group students to foster both cooperative teamwork & individual reflections.	Cooperative work by Piggott (2011) & Shoenfeld's (2016) fourth dimension
	Rules	Guides should outline rules of engagement for each activity, setting participation expectations.	Shoenfeld's (2016) fourth dimension & perspective by Valverde et al. (2002)
Management: Differentiated instruction	Support	Guides should provide strategies for supporting students who may struggle, including scaffolding techniques & additional resources.	Schoenfeld's (2016) third dimension
	Extension	Guides should offer extension activities for students who need additional stimulation to further their understanding.	

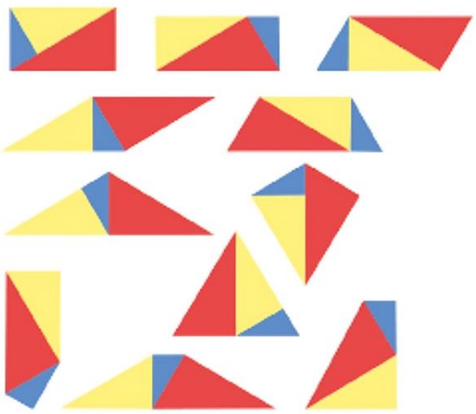
Note. AS: Aspect of segment

Focusing on the quoted example from Innovamat's 4th grade curriculum, if we consider not only the written guide but also the class video introducing the lesson (quoted within the guide and provided through a digital platform), we count 87 different segments in two pages and a three-minute video. Some examples are as follows (**Table 4**).

Once the guide is divided into segments, the tool allows us to characterize each one of them according to our framework on richness. Technologically, the tool was converted into a spreadsheet: each segment of the guide, with its picture, formed a row with several cells to fill out from drop-downs.

Following the described framework and methodology, we firstly identify the type of segment: *instruction*, which requires some actions from the teacher, or *information*, which does not ask for any specific actions. Once the type of segment is identified, we then assess its main contribution to the activity, either to the *task* or to its *management*. If the segment mainly contributes to the task, we assess its cognitive demand, which can be (to) *reproduce*, *relate*, or *reflect*. We also characterize the processes involved, which can include *problem solving*, *reasoning and proof*, *connections*, and *communication and representation*.

Table 4. Examples of segments from guide

Segment from guide: Image	Segment from guide: Text	Short description
Challenge 19 **Convex Polygons** A Special Tangram	Challenge 19, convex polygons, & a special tangram	Number of challenge, title, & subtitle.
0. We hand out a sheet of graph paper, a triangle, or set square, and some scissors to everyone.	0. We hand out a sheet of graph paper, a triangle, or set square, & some scissors to everyone.	Preparatory step involving manipulatives.
It is important to colour in both sides of the pieces, so that the students may turn them around when exploring all the possibilities.	[Tip] It is important to color in both sides of pieces, so that students may turn them around when exploring all possibilities.	Tip for teacher about coloring tangram.
The students may need help when using the triangle and/or set square to draw the perpendicular.	[Tip] Students may need help when using triangle and/or set square to draw perpendicular.	Tip for teacher about some expected difficulties on drawing tangram.
1. We explain to the students that we want to construct all the polygons we can, using the pieces of the Tangram we have made.	1. We explain to students that we want to construct all polygons we can, using pieces of tangram we have made.	Step to start building convex polygons.
A polygon is convex when none of the interior angles are greater than 180 degrees. The students do not need to measure the angles. Explaining this and checking that all the polygons they produce are convex, is enough.	[Tip] A polygon is convex when none of interior angles are greater than 180 degrees. Students do not need to measure angles. Explaining this & checking that all polygons they produce are convex, is enough.	Tip for teacher with a definition for convexity & managing expectations.
Our aim is for everyone to have drawings of the 11 polygons in their logbooks by the end of the session. 	[Tip] Our aim is for everyone to have drawings of 11 polygons in their logbooks by end of session. [Solution with 11 convex polygons]	Tip that includes a picture of 11 solutions.

Additionally, we analyze the contents of the task, considering whether it is *contextualized*, *rigorous*, and *extensible* within the curriculum content blocks (*numbers and operations*, *space and shape*, *measurement*, *statistics and probability*, or *change and relationships*).

If the segment contributes to the management of the activity, we assess the *environment*, considering aspects such as *time and structure*, *grouping*, and *rules*. We also evaluate *differentiated instruction* strategies, looking for *support* and *extension* ones.

To provide a clearer understanding of how our tool is applied in practice, we present a selection of three specific examples from the guide. These examples, shown in **Table 5**, represent a variety of segment types and contributions, and demonstrate how each segment is characterized according to our criteria.

Each column in **Table 5** corresponds to a different segment from the guide, with the rows providing details on their aspects, as described in **Table 5**.

To provide a broader perspective on our analysis, now that we have delved into concrete qualitative examples, we present a quantitative summary of the guide in terms of percentages. It is important to note, however, that this summary, while offering an insightful overview of the distribution of different types of segments in the guide, does not constitute a comprehensive statistical analysis of the tool.

Our primary focus in this study has been on the tool's conceptual development and practical application within a specific educational context.

Table 5. Application of tool to some segments from guide

Segment	[Tip] In event of considering that this activity may take too long, we recommend making them at some point before session.	1. We explain to students that we want to construct all (convex) polygons we can, using pieces of tangram we have made.	[Tip] A polygon is convex when none of interior angles are greater than 180 degrees. Students do not need to measure angles. Explaining this & checking that all polygons they produce are convex, is enough.
Information or instruction?	Information: It provides guidance about managing timing of activity.	Instruction: It provides a clear directive about task.	Information: It provides a content definition.
Task or management?	Management: It offers advice on how to handle timing, rather than describing a specific task.	Task: It outlines a specific task for students to engage in.	Task: It narrows down demand for students.
Task: Cognitive demand	NA	Reflect: This task requires students to apply their understanding of polygons & use problem-solving skills to construct them all with tangram pieces.	Relate: Understanding convexity requires comparing angles.
Task: Processes	NA	Problem-solving: Following a strategy to find them all in a systematic approach; Reasoning & proof: Comparing shapes & finding patterns that facilitate search; & Connections: Relating convexity in polygons with interior angles.	Connections: Relating convexity in polygons with interior angles.
Task: Content	NA	Contextualized: Thanks to video; Rigorous: It covers polygon convexity (4th grade curriculum) within space & shape content block; & Extensible: It can be connected to prior knowledge about shapes & extended to geometric properties & measures.	Contextualized: It is relevant since task requires understanding convexity; Rigorous: It covers polygon convexity (4th grade curriculum) within space & shape content block; & Extensible: It must be connected to prior knowledge about angles.
Management: Environment	Time & structure: It advises on structuring time or class for activity, a key aspect of learning environment.	NA	NA
Management: Differentiated instruction	NA	NA	NA

Consequently, this quantitative summary is intended to supplement our qualitative findings by highlighting patterns or trends in the data (**Table 6**), rather than to assert the tool's reliability and validity across various contexts with empirical certainty. For example, in this case, the quantitative summary informs us that all processes appear in the guide, but the main ones in this session are communication and representation, and problem-solving.

The triangulation process by three math education experts yielded a high degree of consensus, with complete agreement on 80 out of 87 segments, representing almost 90% of the total. The remaining segments presented some discrepancies that can be sorted into three questions that needed to be addressed:

1. How to tag segments that seem to contribute both to the task and its management.
2. How to tag segments that may have different levels of cognitive demand.
3. How to tag segments that involve more than one process or more than one content, or even both.

As described in the subsequent section, discrepancies were addressed through discussions and consensus-building, which contributed to the improvement of the tool, leading to its current version.

Table 6. Quantitative analysis of guide

AS	Aspect options	Quantity	Richness interpretation
Type	Informational	68 out of 87 (78%)	Approximately three informational segments for each instructional one.
	Instructional	19 out of 87 (22%)	
Contribution	Task	42 out of 87 (48%)	There is a balance between segments contributing to task & to its management.
	Management	45 out of 87 (52%)	
Task: Cognitive demand 13 out of 42 (31%)	Reproduce	5 out of 13 (38%)	Just ⅓ of segments involving cognitive demand are reproductive (lower level). Only one is reflective (higher level).
	Relate	7 out of 13 (54%)	
	Reflect	1 out of 13 (8%)	
Task: Processes 18 out of 42 (43%)	• Problem-solving	6 out of 18 (33%)	All four processes are explicitly involved.
	• Reasoning & proof	2 out of 18 (11%)	
	• Connections	1 out of 18 (6%)	
	• Communication & representation	9 out of 18 (50%)	
Task: Content 26 out of 42 (62%)	Contextualized	26 out of 26 (100%)	All content segments are appropriate. They evince that this session was about geometry & measurement.
	Rigorous	26 out of 26 (100%)	
	Extensible	26 out of 26 (100%)	
	• Numbers & operations	0 out of 26 (0%)	
	• Space & shape	19 out of 26 (73%)	
	• Measurement	6 out of 26 (23%)	
	• Statistics & probability	0 out of 26 (0%)	
	• Change & relationships	1 out of 26 (4%)	
Management: Environment 42 out of 45 (93%)	Time & structure	38 out of 42 (91%)	Most of management segments are structural. This seems unbalanced, but we can find an explanation within segmentation process: We considered every single icon or section title (which give structure to guide) as a segment.
	Grouping	3 out of 42 (7%)	There are only three because there are just three moments along session in which grouping is set or changed.
	Rules	1 out of 42 (2%)	There is a lack of explicit segments guiding teacher to construct engagement through participation directives.
Management: Differentiated instruction 3 out of 45 (7%)	Support	3 out of 3 (100%)	All differentiated instruction segments refer to support strategies.
	Extension	0 out of 3 (0%)	There is a lack of explicit segments guiding teacher in how to present more challenging demands to talented students.

Note. AS: Aspect of segment

DISCUSSION

This section starts by describing how discussing the discrepancies during the triangulation process led to the current version of the tool. Later in this section, we expose the limitations of our study.

One such discrepancy arose from segments that seemed to contribute both to the task and its management. For instance, segment 27, 'We hand out a sheet of graph paper, a triangle, or set square, and some scissors to everyone', sparked a debate. One researcher argued that this segment was defining the initial manipulative conditions for the *task*, while the other two researchers viewed the use of manipulatives as more of a *management* issue. This discussion led to the consideration of including a 'use of manipulatives' option within the environment management aspect. But to avoid implying that the use of manipulatives is a condition for richness, we agreed to include this use within the 'time and structure' aspect. Furthermore, we acknowledged that a segment could contribute to both the task and its management, but in such cases, we would focus on its main contribution.

A similar issue arose with cognitive demand. Some task segments may have a cognitive demand that is not entirely of one level, especially considering varying skill levels among students in a class. For example, segment 77, 'We ask students to write names of each one of polygons we have drawn in logbook', could be considered simply reproductive if students already know names, or more about relating if they struggle with some names and have to connect with ideas from the clues to a previous activity. Recognizing this complexity, we agreed to categorize the cognitive demand considering the official curriculum and expected skills for generic students in corresponding grade.

Lastly, there were interesting discussions about whether a task could involve more than one process or more than one content, or even both. This led to a significant change in the tool. Initially, it allowed for only one process or one content to be selected for each task segment. However, this discussion highlighted the fact that the richness from a competency-based perspective lies precisely in the intertwining of multiple contents and processes. Therefore, limiting the options seemed counterproductive. As a result, the current version of the tool allows for the characterization of a segment with multiple processes and contents simultaneously, thereby better capturing the complex richness of the tasks.

When we applied our tool to the guide of a challenge by Innovamat, we confirmed that richness is not a binary attribute (i.e., rich or not rich) but a continuous improvement process (i.e., enrichment). The tool is akin to wear glasses that identify specific areas, where the film (i.e., the guide) is richer and where it could be further enriched.

Our analysis, which combines quantitative data with qualitative insights from each segment, reveals that the specific analyzed guide includes several key traits of richness: it effectively intertwines content and processes, demands a medium-high level of cognitive engagement, and provides explicit management tips, striking a balance between task and management. However, our analysis also highlighted areas for enrichment. The management tips, while useful, lean heavily towards structural guidance. This is partly a result of our segmentation process, which treated every icon or section title as a separate segment, but the guide could benefit from more explicit directives on fostering student engagement and also guidance on presenting more challenging demands to talented students. From our perspective, addressing these areas would enhance the richness of the guide. Moreover, the fact that the application of the tool allows the discovery of areas for improvement to enrich the guide is in itself a new evidence of the utility and reliability of the tool. Obviously, if the guide is eventually iterated to enrich it, it would be interesting to reapply the tool and see how the analysis evolves.

Limitations

Our findings from the triangulation process suggest that the tool we developed works reliably within the specific session guide we analyzed. This robustness indicates that our tool holds potential for broader applications, making it a valuable asset for various stakeholders, including researchers, curriculum designers, and teachers, in analyzing and enriching a wide array of guides or similar documents.

However, we acknowledge some limitations in our study. The first limitation is the absence of a solid statistical analysis and psychometric evaluation of the tool. While our primary focus was on the tool's conceptual development and practical application in a specific educational context, the lack of detailed statistical validation means that we have not quantified the tool's psychometric properties, such as its reliability and validity across different contexts. This omission limits our ability to assert the tool's universal applicability and effectiveness with empirical certainty. Future research should include a comprehensive statistical examination and psychometric testing to establish the tool's robustness and ensure its utility and trustworthiness in a wider range of educational settings.

Another limitation is that our study applied the tool to only one guide, raising questions about its adaptability and effectiveness across different types of session guides or educational materials with varying structures and content. Although the tool demonstrated efficacy in our specific context, its performance across various educational levels or cultural backgrounds remains untested. Future research should aim to apply the tool to a diverse range of session guides to evaluate its versatility rigorously and make necessary adjustments for broader usability.

Additionally, our analysis treated the session guide as a static document and did not extend to its practical application in classroom settings. We did not examine how teachers interpret and utilize the guide in real-time, nor how their individual characteristics, such as mathematical knowledge, pedagogical skills, and classroom management abilities, influence the implementation and effectiveness of the guide. Investigating the dynamic aspect of how guides are employed in educational settings is a crucial area for future research.

Finally, the framework we used to define richness in educational activities could be expanded. Our analysis primarily focused on certain key aspects, but there are other dimensions that could significantly contribute to the concept of richness in math education. These include the nature of activity assessment and the emerging field of socioemotional skills in mathematics education, as highlighted by recent literature and institutions like OECD (2017b). Expanding the scope of future iterations of our study to incorporate these additional dimensions would not only enrich our understanding of richness from a competency-based perspective but also align our framework with the evolving dynamics of contemporary educational discourse.

CONCLUSIONS

Given the pivotal role of resources and documents in shaping learning experiences in education, a considerable body of research has been dedicated to exploring their impact (Trouche et al., 2018). Teaching guides, in particular, serve as a critical link between curriculum design and classroom implementation, significantly influencing teacher performance and serving as both an operational tool for implementing activities and a resource for continuous professional development. Despite the extensive literature on textbooks (Gueudet et al., 2013), we found a noticeable gap in the research when it comes to analyze teaching guides from a competency-based perspective.

Our study aims to bridge this gap, offering a new perspective on teacher guides analysis and improvement. By integrating theoretical frameworks, we provide a comprehensive tool for educators and curriculum designers to enhance the richness of math teaching guides.

In defining the concept of a teaching guide within our study, we distinguish it from traditional textbooks by applying DAD. Unlike textbooks, which are seen as resources, teaching guides are considered as documents, encompassing not just tasks, but also management. By analyzing teaching guides from a competency-based perspective, we explore their potential richness in terms of content, processes, competencies, and classroom management, which are often overlooked in traditional textbook analysis. This approach allows us to assess the intrinsic quality of a guide as a document, factoring in its mathematical, didactic, and user experience aspects, and setting stage for a more nuanced understanding and enhancement of teaching resources in mathematics education.

Central to our framework is the concept of rich math activities, viewed through a competency-based lens drawn from Niss and Højgaard's (2019) definition of mathematical competence, the powerful class characterization by Schoenfeld's (2016) TRU-math (2016), and de Lange's (1999) work on cognitive demand. Furthermore, our exploration into the traits of rich math activities has led to the formulation of a framework that incorporates several key components: the promotion of mathematical processes, the contextualization and rigor of content, the cognitive demand of tasks, differentiated instruction, and the creation of an engaging and autonomous learning environment. Our framework offers a holistic perspective on guide evaluation and improvement, contributing to the ongoing discourse on math guides design and providing educators and curriculum designers with a solid tool to enhance the richness of their guides.

The development of the tool that characterizes segments of a teaching guide according to this preliminary exploration of richness stands as a main contribution. The tool was tested and improved through a triangulation process involving three math education experts. Such discussions led to significant changes in the tool, such as allowing for the characterization of a segment with multiple processes and contents simultaneously, thereby better capturing the complex richness of the tasks. Applying the framework to a specific guide revealed that while the guide was rich in many aspects, there were areas for potential enrichment. This suggests that our tool provides a valuable framework for identifying areas of improvement in teaching guides, but it also underscores the need for continuous refinement and expansion of the tool itself.

A last significant finding emerged from our study: the analyzed guide exhibits traits of richness, yet also reveals areas that could be further enriched. This not only evidences effectiveness of our tool in characterizing richness traits but also highlights its utility in suggesting areas of improvement. Specifically, the guide could benefit from enhancements in fostering student engagement and presenting more challenging demands to talented students. This finding presents a valuable opportunity to refine guide in these specific aspects.

While our developed tool shows promise in analyzing and enriching teaching guides, our study acknowledges some limitations. Firstly, the absence of a detailed statistical analysis and psychometric evaluation means we have not fully established the tool's reliability and validity across different contexts, which is crucial for asserting its universal applicability. Secondly, our application of the tool to only one specific guide raises questions about its adaptability. This limitation underscores the need for future research to apply the tool to diverse session guides and contexts. Additionally, our study did not explore the dynamic use of the guide in classroom settings, including how teachers' characteristics might influence its effectiveness, an important area for subsequent investigations. This endeavor aligns with the learning through teaching approach (Leikin & Zazkis, 2010), emphasizing the learning that occurs for teachers during the act of teaching, and the potential of teaching guides to facilitate this learning. Lastly, the framework used to define the concept of 'richness' in educational activities could benefit from incorporating emerging fields such as socioemotional skills in mathematics, which would provide a more comprehensive understanding of richness in math education and align our framework with other contemporary educational trends.

In conclusion, our study contributes to the ongoing discourse on math guides design by providing a nascent tool for educators and curriculum designers to enhance the richness of their guides. It also underscores the importance of continuous improvement in the design of teaching guides. The richness of activity is a necessary condition for the richness of the guide, but a rich activity does not automatically translate into a rich guide. Just as a filmmaker continuously refines their craft to better capture and communicate the richness of reality, educators and curriculum designers should strive to enhance richness of their guides to better facilitate learning experiences for both students and teachers.

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