MODESTUM EURASIA Journal of Mathematics, Science and Technology Education, 2024, 20(10), em2514
ISSN:1305-8223 (online)
OPEN ACCESS Research Paper https://doi.org/10.29333/ejmste/15198

A didactic proposal for teaching factorization cases of expressions of the form $ax^2+by+cx+ay+exy+f$ through Mathigon

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Received 28 August 2023 ▪ Accepted 29 January 2024

Abstract
This paper presents the results of research conducted in a classroom setting with eighth-grade students in a public school in Argentina. The experience aimed, within the framework of an algebra course, to introduce the design and results of the implementation of a didactic proposal designed to establish relationships between expressions of the form $ax^2+by+cx+ay+exy+f$, with $a, b \in N$ and $c, d, e, f \in Z$, as a product of factors through the virtual manipulative Mathigon. The evidence collected leads to the identification that algebraic work using virtual manipulatives allows students to give a different meaning to mathematics, explore various forms of representation, and construct their own knowledge.

Keywords: Mathigon, mathematics education, factoring, algebra, didactic engineering

INTRODUCTION

In recent years, and even more so since the experience during the times of the pandemic, there has been a notable interest in the adoption of virtual resources in education (Salas et al., 2021). Mathematics has not been an exception to this rule, as it has been a source of inspiration for the design and application of various virtual educational materials, aimed at enabling students in non-presential contexts to engage with various concepts through manipulation, while promoting opportunities that emphasize cognitive processes (Vergara, 2021).

However, the constant evolution in science and technology has raised the importance of incorporating active methodologies supported by technological tools, especially in the field of mathematics education in the current digital era (Hodovaniuk et al., 2020). Therefore, considering today's dynamic and advanced environment, the ability to adapt quickly to developing technologies and modern management methods is essential for the future success of students, where those who possess a solid mathematical knowledge and the ability to apply mathematical skills will demonstrate more successful performance in various areas, from education to future careers (Kliziené et al., 2021).

In this context, the critical importance of mathematical literacy as an essential competency for students to be able to function effectively in their daily lives and in future educational and occupational settings is underscored (García et al., 2017). Therefore, the teaching of mathematics in secondary education emerges as a fundamental pillar to lay the solid foundations of mathematical literacy, where it is not limited only to the acquisition of knowledge but encompasses a deep understanding and effective application of mathematical principles (Armando et al., 2023).

Both the design and selection of digital resources have received special attention in the teaching and learning processes in the classroom, as Chiappe (2016) suggests that digitally designed educational content adequately addresses current technological, communicative, and pedagogical challenges. Therefore, thematic updating extends beyond the classroom, as it allows the incorporation of innovative methodologies and strategies for a modern society. This may be the reason

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Contribution to the literature

- The article contributes to educational literature by introducing an innovative perspective in algebra education, offering new ideas and approaches to address this discipline.
- The researchers emphasize the relevance of digital tools in the educational process, especially in the field of algebra, highlighting the need to integrate technologies such as virtual manipulatives to enhance students' understanding of complex concepts.
- The research contributes to the educational field by promoting the exploration of representations in algebra teaching, highlighting the continuous growth in the educational field and suggesting that these innovations can have a lasting impact on the way they are approached.

why Cabero and Marín (2014) refer to the importance of the internet as a knowledge construction stage, as it has evolved from being a repository of information to becoming a social instrument for creating new learning.

In this regard, research such as that reported by Mato-Vázquez et al. (2018) states that there are at least three types of virtual materials: reference material, digital textbook, and digital didactic material. Within digital didactic material, we recognize the existence of applications or specialized software, as well as platforms, which bring with them the possibility of exploration through the so-called virtual manipulatives, simulating concrete scenarios in a 100% virtual environment, which, according to our experience, prove to be a powerful resource for teaching and learning various mathematical concepts, including algebra (Babo et al., 2023).

This is why this article begins by recognizing the potential offered by virtual manipulatives in teaching and learning, especially in the development of algebraic skills, particularly the connection between the area and the expression of some polynomials (Huda, 2019). The objective of this work is to introduce the design and results of the implementation of a didactic proposal designed to establish relationships between expressions of the form $ax^2+by+cx+ay+exy+f$, with $a, b \in \mathbb{N}$ and $c, d, e, f \in \mathbb{Z}$, as a product of factors through the virtual manipulative Mathigon. In particular, it seeks to simplify to the minimum the factors whose product determines the polynomial representing the area of a rectangular region formed by tablets represented through the Mathigon virtual manipulative (Lee et al., 2023). To present the proposal, the structure guiding the research development is presented in four moments, namely: the justification of the design, the theoretical framework that supports the design, the methodological proposal, analysis, and the discussion (Palma et al., 2023).

THEORETICAL FRAMEWORK

As mentioned in the introduction, this theoretical framework serves as a guide for the proposal whose objective is to introduce the design and results of the implementation of a didactic proposal for teaching factorization cases of expressions of the form $ax^2+by+cx+ay+exy+f$, with $a, b \in \mathbb{N}$ and $c, d, e, f \in \mathbb{Z}$, as a product of factors.

Theory of Didactic Situations in a Teaching Model with Virtual Manipulatives

The theory of didactic situations (TDS) is a methodology developed by the mathematician and educator Brousseau (1986). This theory proposes an approach to teaching mathematics based on the construction of knowledge through concrete and meaningful situations.

TDS should be seen as a creation process, designed to provoke obstacles in the student that will allow them to build their own knowledge (López-García, 2023). For this, it is necessary to consider the level of knowledge of the students, as well as their way of acting and thinking, which can be naturally manifested when using virtual manipulatives in teaching (Rumanová, 2022). This theoretical concept addresses the ways in which interlocutors in a teaching-learning relationship develop specific mathematical knowledge as they interpret the intentions and expectations of the other during the communication process (Rahayu et al., 2020).

TDS proposes a teaching model in which the teacher prepares an adidactic situation, provides the problem and the tools to the students, and guides their interaction with the tools (da Silva & Duarte, 2019). The adidactic situation leads students to construct knowledge related to what is being taught, so the didactics experts design the adidactic situation, and the teacher appropriates it by preparing their intervention during the adidactic situation and observing the students' interaction with the tools (Zaikin et al., 2017). After the adidactic situation, the teacher organizes a sharing session to verify that the students have constructed the expected knowledge and finally institutionalizes the knowledge by relating it to the knowledge constructed during the adidactic situation (Tetchueng et al., 2007). This way, learning through adaptation is consolidated, resulting from a personal and contextualized experience that, in a scenario mediated by manipulatives, becomes a source of meaning for objects (Oliveira Jr & Datori Barbosa, 2023).

Virtual Manipulatives

Virtual manipulatives are digital tools that enable students to manipulate objects, explore, and visualize abstract concepts interactively, potentially enhancing their understanding, construction, and retention of information (Martín Díaz et al., 2023). One of the many possibilities offered by working with virtual manipulatives has been reported by authors such as Mato-Vázquez et al. (2018), Guizado and Ortiz (2022), and Chacón et al. (2023). They assert that the adaptability provided by these resources allows instruction to be tailored to the individual needs of learners in various applications, systems, and contexts. According to Caro (2015), this adaptability enables:

- The application of the resource at any educational stage.
- An increase in student motivation and engagement.
- The facilitation of greater student attentiveness.
- A more effective response to student needs.

In addition to the aforementioned benefits, it is recognized that virtual manipulatives offer the opportunity to manipulate objects that are often impractical due to issues of dimension, practicality, or economic constraints (Moral-Sánchez et al., 2022). Among their benefits are, as follows (Martin et al., 2022):

1. **Enhanced conceptual understanding:** Virtual manipulatives allow students to visualize abstract concepts, potentially improving their understanding. For example, a student can manipulate a virtual model of a cell and observe how its parts function, leading to a better comprehension of cell structure and function.
2. **Promotion of active learning:** Virtual manipulatives enable students to actively engage in their own learning by experimenting with objects and observing their interactions. This promotes a more participatory approach to learning and helps students retain information more effectively.
3. **Increased motivation:** Virtual manipulatives can make learning more interesting and appealing to students, potentially boosting their motivation. Students may become more engaged in the learning process and more willing to participate in activities.
4. **Improved information retention:** Virtual manipulatives can assist students in retaining information more effectively, as they can experiment with objects and visualize concepts. This may make information more memorable and easier to recall over the long term.
5. **Facilitation of collaborative work:** Virtual manipulatives can be employed in group activities, allowing students to collaborate in problem-solving and concept exploration. This fosters collaboration and the exchange of ideas, which can enhance comprehension and learning.
6. **Immediate feedback:** Virtual manipulatives can provide instant feedback to students, enabling them to correct their errors and improve their understanding. For instance, when learning about fractions, a student can use a virtual manipulative to observe how fractions are divided and added, receiving immediate feedback on the correctness of their operations.
7. **Personalized learning:** Virtual manipulatives can be customized to meet individual student needs. For example, if a student struggles with a particular concept, they can use a specific virtual manipulative tailored to that concept and work with it until they fully understand it.
8. **Promotion of technological skills:** The use of virtual manipulatives can help students develop technological skills essential in today's world. By using digital tools in their learning, students can enhance their ability to navigate technology and work more effectively in digital environments.

In conclusion, virtual manipulatives offer numerous benefits in middle education, including enhanced conceptual understanding, the promotion of active learning, increased motivation, improved information retention, facilitation of collaborative work, provision of immediate feedback, support for personalized learning, and promotion of technological skill development (Chacón-Castro et al., 2021).

Although the use of virtual manipulatives did not emerge solely to address the needs of virtual education during the pandemic, its popularity did surge in this context; one digital platform with extensive exploratory possibilities in mathematics is Mathigon (Sepeng, 2023). As its designers assert, Mathigon is founded on constructivist principles and aims to provide students with the opportunity to explore, construct, and question their knowledge through a virtual environment (Keldgord & Ching, 2022).

Mathigon as a Design Manipulative

Mathigon is an online learning platform that offers a wide range of interactive tools and educational resources for teaching and learning mathematics (**Figure 1**).

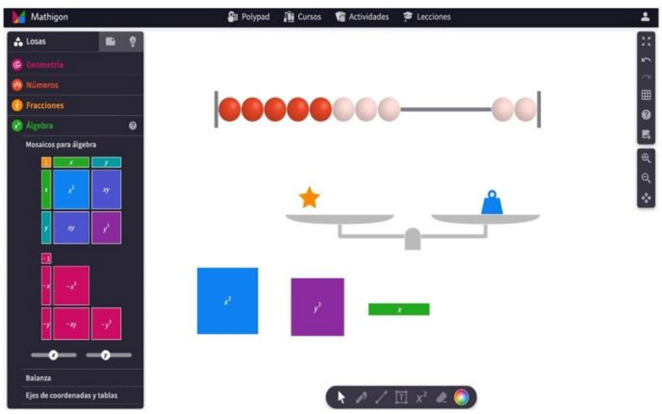


Figure 1. Mathigon platform interface ("Polypad – virtual manipulatives – mathigon", 2023)

Among the tools offered by the platform for algebra work, the following can be highlighted (Martín-Gutiérrez et al., 2017):

- **Interactive manipulatives:** Mathigon provides numerous interactive manipulatives that allow students to experiment with algebraic concepts practically. For instance, there are manipulatives for solving equations, graphing functions, factoring polynomials, and performing matrix operations.
- **Interactive algebra courses:** Mathigon offers interactive algebra courses that guide students in learning both basic and advanced algebraic concepts. These courses include a combination of theoretical explanations, solved examples, and interactive exercises.
- **Symbolic algebra tool:** Mathigon provides an online symbolic algebra tool that enables students to perform algebraic operations with variables and symbolic expressions. The tool includes functions for simplifying expressions, factoring polynomials, solving equations, and graphing functions.
- **Graph explorer:** Mathigon features a graph explorer that allows students to graph functions and experiment with their transformations. Students can adjust function parameters to observe how the graphs change, enhancing their understanding of algebraic concepts.
- **Explanatory videos:** Mathigon offers an extensive collection of explanatory videos covering a wide clear explanations.

In this way, algebra significantly benefits from the use of Mathigon, which, by employing diverse resources such as tiles, tables, axes, and balances (**Figure 2**), leads students through various forms of representation. This approach makes the consolidation of a concept a construction process that addresses much deeper issues than mere procedural mechanization (Thompson et al., 2022).

The richness of these inputs in an algebraic context is enhanced when students, through manipulation, can develop a series of conjectures that result in logical reasoning based on thought processes accompanying the construction of algebraic notions (Mayasari et al., 2020).

In this line, the work presented by Ferreira and Mendes (2020) highlights how the exploration of first grade undergraduate students through the Mathigon platform led to a greater understanding of the idea of relating verbal (rhetorical), symbolic and algebraic representation systems. Likewise, the authors of this article have experienced how practical manipulatives, especially in topics related to problem solving, are constructed in a conventional manner, without taking advantage of the capability of online technology. Therefore, they proposed the creation of an online and open-source repository of interactive materials using the Mathigon tool, which facilitated student interaction, motivation and interest.

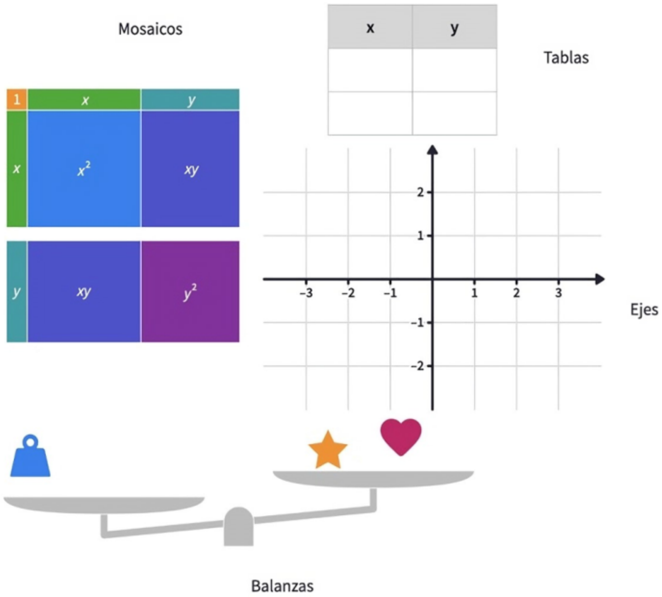


Figure 2. Tools offered by Mathigon for algebra work ("Polypad – virtual manipulatives – mathigon", 2023)

Similarly, Kechil et al. (2022) present a study focused on undergraduate mathematics students, where they were evaluated under control methodology, with the objective of developing an active methodology that provides students with an educational and interactive experience through the use of digital tools such as PowerPoint and Mathigon.

Regarding the results, the authors demonstrated that the application of these dynamics allowed students to improve in the understanding of geometric concepts, where it was recognized that digital tools promote active learning and improve results in mathematical subjects.

In this way, we recognize virtual manipulatives as an extension of concrete manipulatives. From this perspective, and even more so from experience, virtual manipulatives are seen as a pathway for exploring and constructing mathematical ideas and concepts, motivating students to explore without limitations.

This work aims to design a learning sequence that seeks to recognize, through the voices of both the implementing teacher and the students undergoing the process, the benefits that virtual manipulatives bring to the development of algebraic skills. The following section presents the methodology employed for the design.

METHODOLOGY

The didactic engineering is assumed as the basis for the methodological structure of the class intervention, which proposes the development of four phases or moments:

1. Preliminary analysis
2. Conception and a priori analysis of didactic situations
3. Experimentation

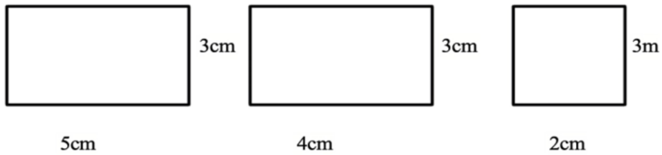


Figure 3. Rectangular arrangements-1 (Source: Authors' own elaboration)

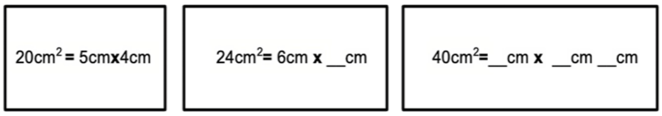


Figure 4. Rectangular arrangements-2 (Source: Authors' own elaboration)

4. Posterior analysis and evaluation

The selected population for this research consists of 32 ninth-grade students from the Mercedes Pacheco Institution, with 20 female students and 12 male students, aged between 14 and 16 years old.

Additionally, a non-probabilistic random sample of 43 students from the public institution was considered for the research.

Design of Didactic Engineering

Preliminary analysis

This preliminary analysis involves recognizing the starting point of the students. In our case, it began with a pre-test aimed at determining the extent to which students were familiar with issues related to area, perimeter, meanings of variables, variable operations, among others. The test consisted of four questions, which were validated by an expert. Below are the pre-test questions.

Pre-test

1. Calculate the area of the rectangular arrangements in **Figure 3**.
    a. How did you do it?
    b. Why did you do it the way you did?
2. What are the measurements of the sides of the rectangular arrangements in **Figure 4**?
    a. How do you justify your answer?
3. Look at the representations in **Figure 5**. What meaning do you attribute to 'x'?
4. What is the result of the following operations? Why? Justify your answer.
    a. 2+x+4+x+5+x=
    b. (4y)(2x+2y)=
    c. 5+x=

Conception & a priori analysis

This a priori analysis included a predictive part that focused on characterizing the didactic situation to be proposed to the students.

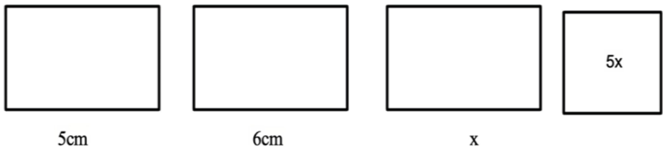


Figure 5. Rectangular arrangements-3 (Source: Authors' own elaboration)

In our case, we recognize this situation as a sequence consisting of four phases: motivation and presentation of the tool phase, conceptualization phase, exercise with manipulative phase, and evaluation phase.

Below, we present how each of these phases is assumed in this research.

- **Motivation and presentation of the tool phase:** The teacher presents the students with the importance of the topic and shows them some examples of situations in which it is used. The teacher explains how the pedagogical tool that has been used is useful for addressing problems in a general manner.
- **Conceptualization phase:** The teacher explains the theory behind the topic to be developed. Specific cases that have been studied are explored, and practical examples demonstrating their application are presented.
- **Exercise phase:** Students work in pairs or small groups, based on their prior knowledge of the topic. The teacher circulates around the classroom to observe students' work and answer questions. These exercises are progressive and allow students to practice factorizing expressions. Teamwork and group problem-solving are encouraged.
- **Evaluation phase:** Student progress is assessed through the resolution of practical exercises and observation of their participation in group activities. Students are invited to share their strategies and solutions for factoring expressions. The teacher summarizes concepts and strategies learned during the session and emphasizes the importance of factorization in solving mathematical problems. Students are encouraged to reflect on what they have learned and to raise any questions or doubts they may have.

Experimentation

This is the phase of didactic engineering, where the didactic sequence was implemented in six sessions. The sessions were distributed into three moments each, with two classes of two hours each. The phases mentioned in the previous section are divided into the two classes implemented in week.

Table 1 shows implementation of phases according to the week of cases (equivalences).

Table 1. Implementation of phases according to the week of cases (equivalences)

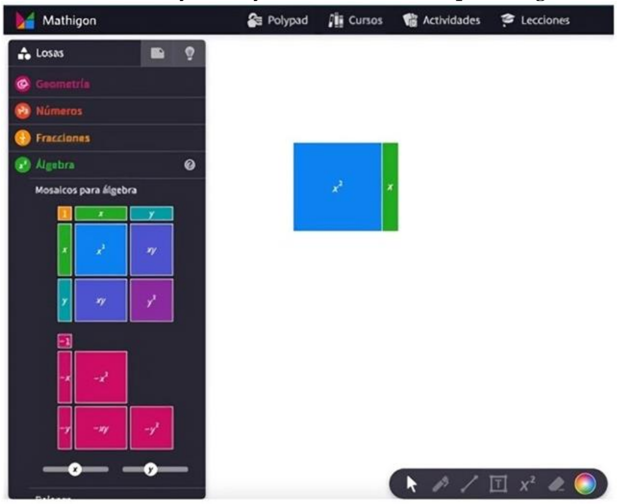
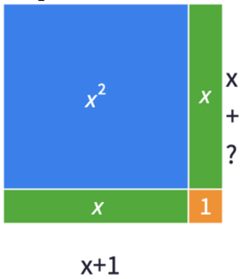
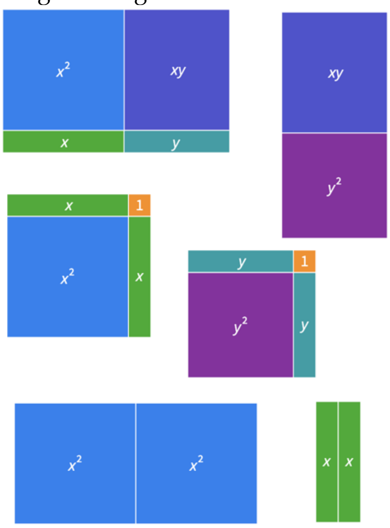
Week	Class	Explanation
1	1	**Phase of motivation & exploration of tool:** As an initial motivation strategy, an exercise of material manipulation was conducted, which began by providing instructions & examples for identifying positive terms of a polynomial. Instructions were given regarding placement of tiles, for example, emphasizing that tiles could only be placed consecutively if they shared sides of equal length.  Interfaz Mathigon polypad Students were informed that primary purpose of each arrangement was to create rectangles in such a way that there were no empty areas inside them. After creating rectangle, they needed to correctly determine lengths of sides required, which are typically base & height. To achieve this, length of each "segment" forming total side was added up, as shown in example.  Mathigon array **Phase of exploration with manipulative:** Students were asked to create at least two versions of rectangular arrays with sides equivalent to: **x+1, x+y, 2x+3y, x+y+2, 2y+1,** & **x+2**
	2	**Phase of evaluation:** Teacher presented students with a series of rectangular arrays created with Mathigon in which students had to recognize lengths of their sides & find other equivalent ones.  Mathigon array **Closing phase:** Teacher summarized concepts & strategies learned during session.

Table 2. Implementation of phases according to the week of cases (areas)

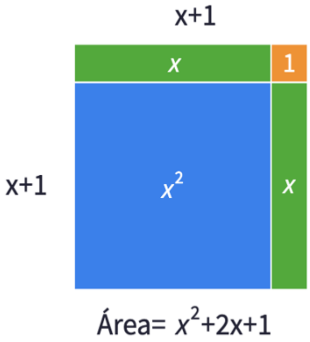
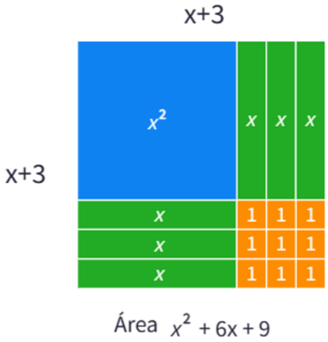
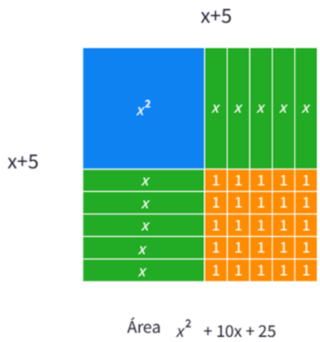
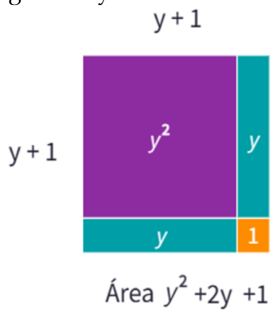
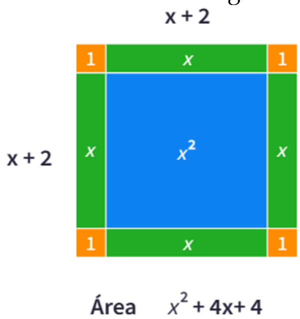
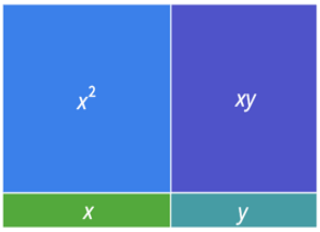
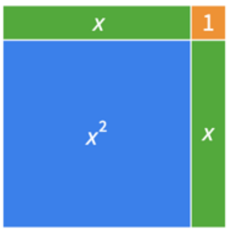
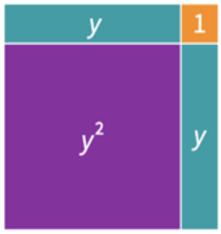
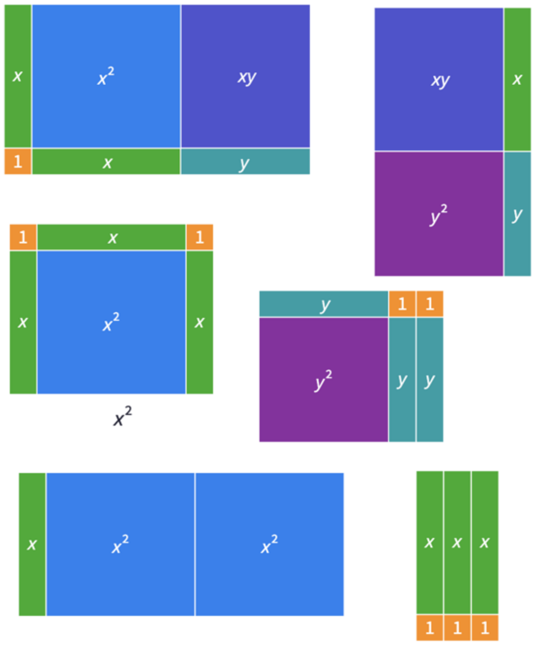
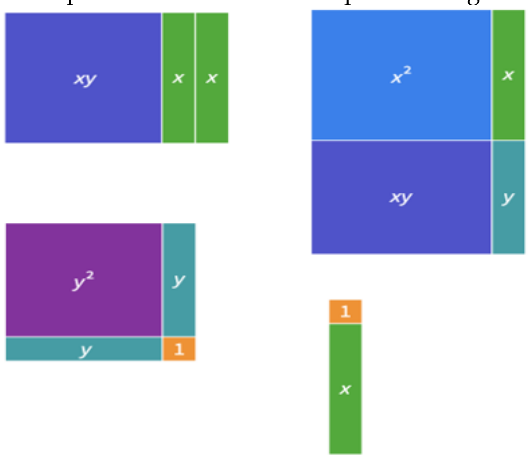
Week	Class	Explanation
2	1	**Phase of motivation:** Students were asked to recall what was covered in the previous session and create at least two versions of rectangular arrangements with sides equivalent to: **2x+3, 2x+y, & x+3y**. **Phase of conceptualization:** In this exercise, tiles were used to represent polynomials of form $ax^2+by+cx+ay+exy+f$, as presented below.  Mathigon array-1  Mathigon array-2  Mathigon array-3  Mathigon array-4  Mathigon array-5 Phase of evaluation: Students were presented with a series of figures created with Mathigon that they had to reconstruct & find areas & dimensions for. They were also asked to create other figures.  Mathigon array-1  Mathigon array-2  Mathigon array-3  Mathigon array-4  Mathigon array-5  Mathigon array-6
	2	Teacher proposed a series of rectangular arrangements made with Mathigon in which students had to recognize lengths of their sides & find equivalent ones. **Closing phase:** Teacher summarized concepts & strategies learned during session.

Table 2 shows implementation of phases according to the week of cases (areas).

Table 3 shows implementation of phases according to the week of cases (reducible factors).

Table 3. Implementation of phases according to the week of cases (reducible factors)

Week	Class	Explanation
3	1	**Phase of motivation & exploration of tool:** Students were asked to recall what was covered in previous session. They were required to recognize lengths of sides of rectangular arrangements & find equivalent ones .  Mathigon arrays **Phase of conceptualization:** First, polynomial **x^2+5x+6** was presented using tiles. Then, it was shown how to find an algebraic expression in form of a product of factors that is equivalent to represented polynomial. Students were tasked with working on following examples: **$x^2+7x+12$**, **$2x^2+5x+2$**, **x^2+6x+9**, & **$3x^2+9x+6$**.
	2	**Phase of evaluation:** Several polynomial examples were presented to students, which they had to represent both through manipulative & from their general & reducible algebraic expressions. They were asked to represent area & factored expression of given representations.  Mathigon arrays Represent graphically by indicating dimensions of rectangular arrangement: **x^2+9x+6** & **$2x^2+4x+4$**. **Closing phase:** Teacher summarized concepts & strategies learned during session.

Post-analysis & validation

Following experimentation phase is the post-analysis phase involving analyzing data collected throughout experimentation, including observations of teaching sequences and student productions in or out of class.

Techniques

According to this section, the development of the research considered didactic educational evaluation and documentary analysis, enabling consecutive student participation with the support of instructional material.

According to this section, the development of the research considered didactic educational evaluation and documentary analysis, with the aim of encouraging consecutive student participation through the use of material, based on different theories.

TDS propose a teaching approach based on the construction of knowledge through concrete and meaningful situations.

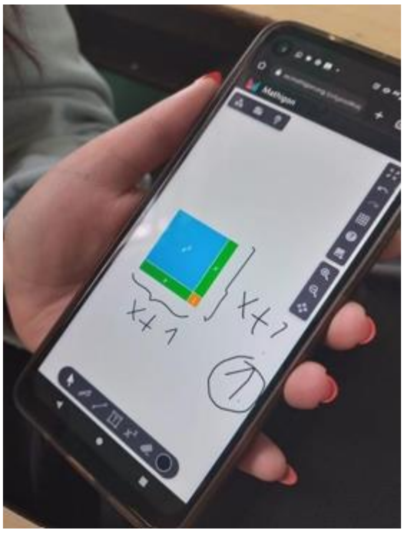


Figure 6. Initial exploration of some of the tools (Source: Authors' own elaboration)

TDS focuses on the interaction between the teacher and students during the communication process, where specific mathematical knowledge is developed as intentions and expectations of each other are interpreted. Therefore, the use of virtual manipulative tools that allow students to manipulate objects, explore, and visualize abstract concepts interactively was proposed. In this case, Mathigon was used as the tool, being an online learning platform that offers a wide variety of interactive tools and educational resources for teaching and learning mathematics.

RESULTS

Below are the results obtained during each of the implementation phases.

Results Obtained During Implementation of Phases by Week

Week 1

Class 1: During the class, the teacher introduced the students to a virtual tool called Mathigon, which would be used as a resource for learning algebra. She explained in detail how to install the application and how to use Mathigon Polypad's algebra tiles, providing students with their first exposure to the topic.

Since it was the students' first time encountering this tool, an initial exploration of some of the tools available in the application was conducted. The teacher guided the students step by step, showing them how to use the basic functions and tools of Mathigon Polypad.

Since it was the students' first time encountering this tool, an initial exploration of some of the tools available in the application was conducted (**Figure 6**). The teacher guided the students step by step, showing them how to use the basic functions and tools of Mathigon Polypad. This introduction to Mathigon and algebra tiles

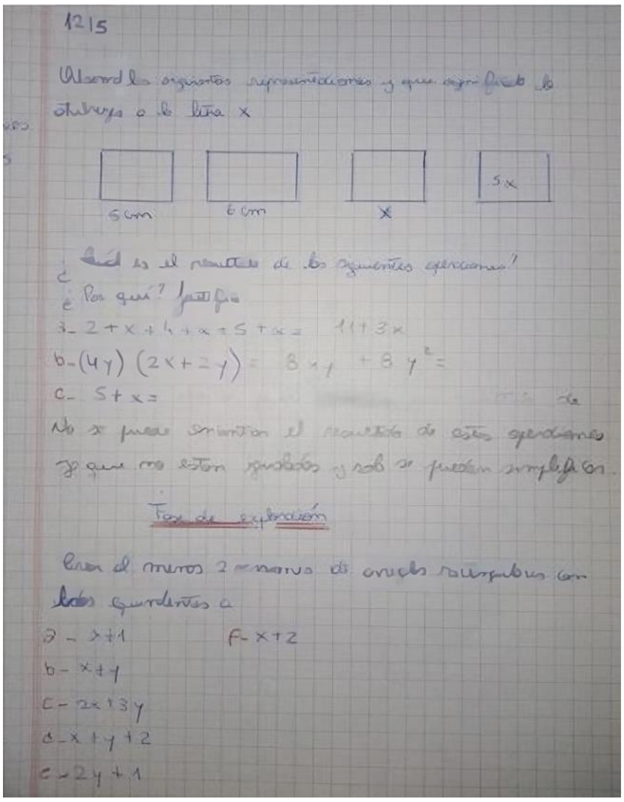


Figure 7. Students' performance in pre-test-1 (reprinted with permission of the student)

provided the students with a new perspective on learning mathematics. It allowed them to visualize abstract concepts in a concrete and manipulable way, facilitating their understanding and promoting algebraic reasoning.

It is important to highlight that this initial experience with Mathigon lays the foundation for future, more advanced explorations and applications of the tool. Finally, from this first encounter, it is noteworthy that the students demonstrated a high level of participation, actively engaging with questions and ideas for the use of digital resources. Furthermore, it was found that the use of the tool allowed them to innovatively explore the different instructions for tile placement. This demonstrated that the students visualized the concepts concretely when they successfully created rectangles without leaving empty areas within them. In the same way, they were able to determine the correct length of the sides of the rectangles, emphasizing that the use of the Mathigon tool helped transfer the students' foundational knowledge and apply it in real situations, demonstrating deep understanding and readiness to apply their knowledge in algebraic problem situations.

Class 2: In the pre-test, the students demonstrated excellent performance and mastery of the topic of area formulas for geometric figures (**Figure 7**). They did not encounter difficulties in recalling the formulas and were capable of recognizing the measurements of missing sides or determining whether the unknown referred to a side or the area of the figure.

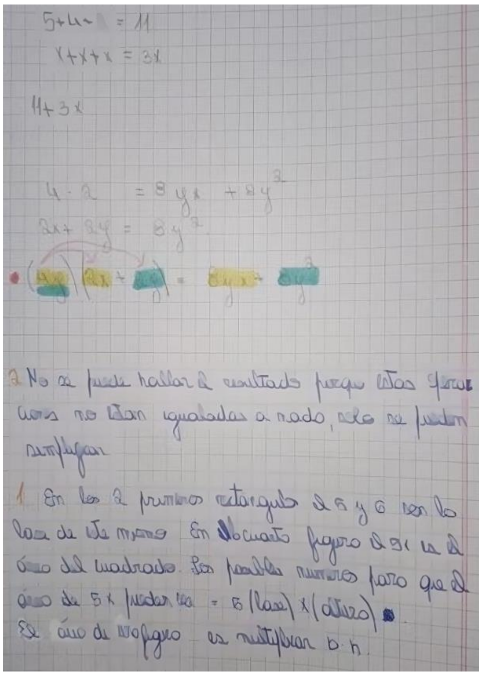


Figure 8. Students' performance in pre-test-2 (reprinted with permission of the student)

What stood out from these results was their ability to analyze and understand the rationale behind the formula for the area of a rectangle. The students could reflect upon and provide well-founded explanations for why multiplication of the base by the height is used to calculate the area of this specific geometric figure.

This result was highly encouraging as it indicated that the students had internalized and developed a deep understanding of the formulas and their application. Their capacity to write an analysis of the rectangle's area formula showcased critical thinking skills and a solid grasp of the subject matter (**Figure 8**).

Based on the results obtained from the pre-test, it was evident that the students possessed the necessary skills to solve problems related to area, perimeter, variable meanings, variable operations, among others proposed in the initial phase. This allowed them to satisfactorily complete the questionnaire. Thus, the results indicated that the students were prepared and ready to advance in their learning of new concepts and challenges in the field of geometry and area calculations (**Figure 9** & **Figure 10**).

During activity 4, some difficulties were observed regarding two specific concepts.

Firstly, some students had trouble understanding the combination of like terms. This refers to their ability to add the numerical coefficients and count the variables correctly that appear in an algebraic expression. For example, in the expression 2+x+4+x+5+x.

On the other hand, other students encountered difficulties in comprehending the distributive property. This difficulty manifests when multiplying the numerical coefficient by each term within parentheses. It can be attributed to a lack of understanding of the concept, confusion with signs within multiplication, or insufficient practice and exercises. For example, in the expression (4y)(2x+2y), the coefficient 4y must be multiplied by each term within the parentheses, i.e., by 2x and by 2y. Some students might make errors when performing this multiplication correctly, which can impact the final result of the expression.

It is important to note that these difficulties are common in the process of learning algebra and required specific attention from teacher (**Figure 11** & **Figure 12**).

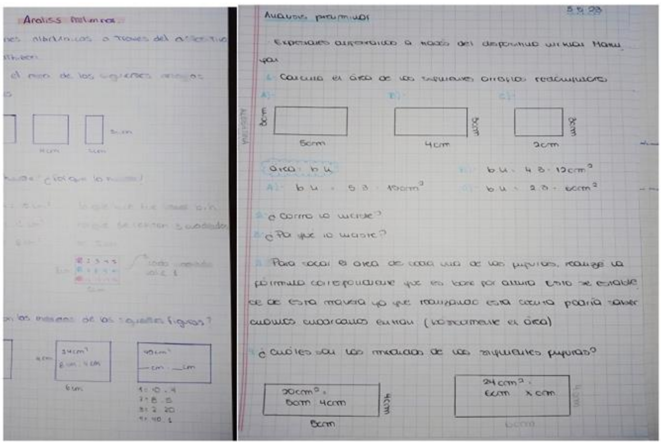


Figure 9. Students' performance in pre-test-3 (reprinted with permission of the student)

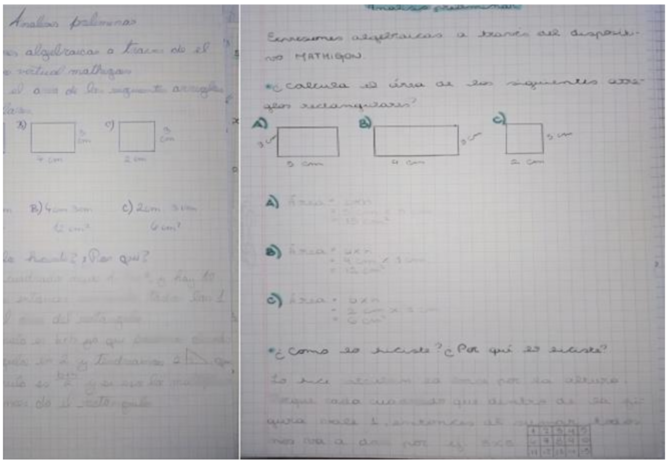


Figure 10. Students' performance in pre-test-4 (reprinted with permission of the student)

Week 2

Class 3: According to what the students expressed, this was the activity that represented the greatest challenge. To carry it out, the children needed to ask questions that would give meaning to the variable and, in particular, to the value it represented. To be certain about the impossibility of a numerical assignment, it was necessary to arrange the algebraic tiles in a way that manual comparison allowed them to recognize the absence of a pattern of a possible unit of measurement. This exercise led to a clearer path for the students

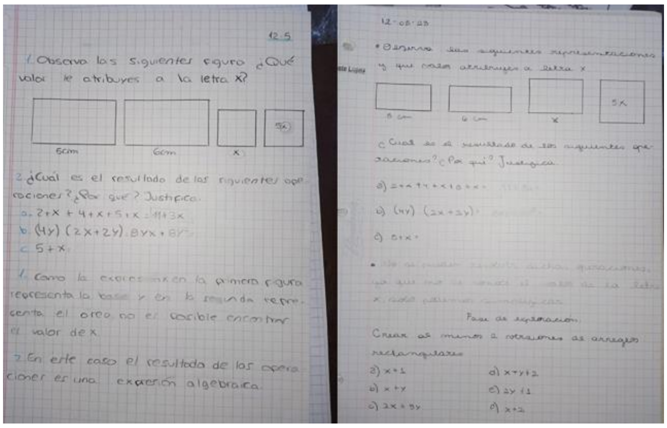


Figure 11. Example-1 (reprinted with permission of the student)

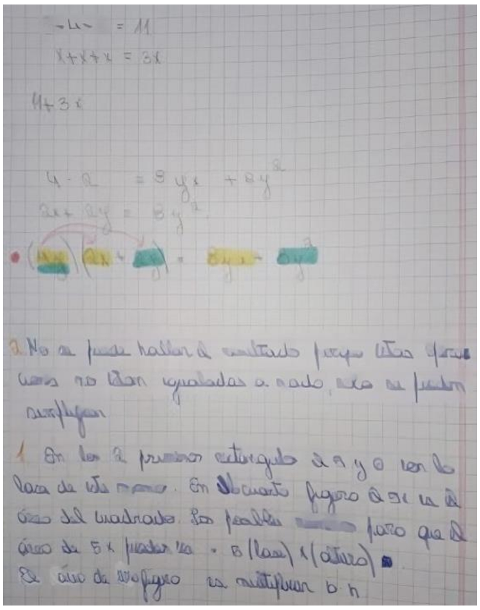


Figure 12. Example-2 (reprinted with permission of the student)

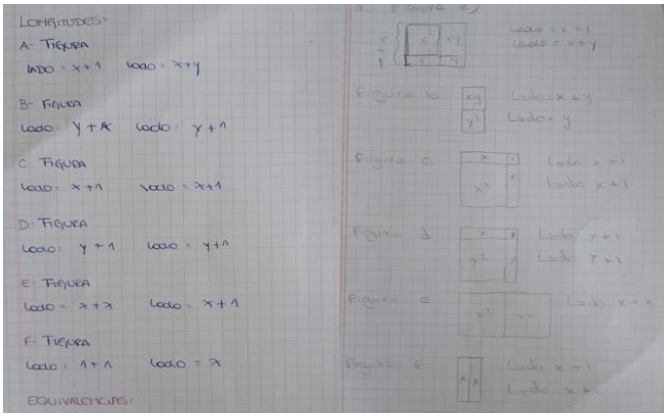


Figure 13. Example-3 (reprinted with permission of the student)

regarding the transition from interpreting the evaluated letter to assigning it a numerical value–in this case, 1.

It recognized that this represents a specific but unknown number that can be operated on directly, yet it

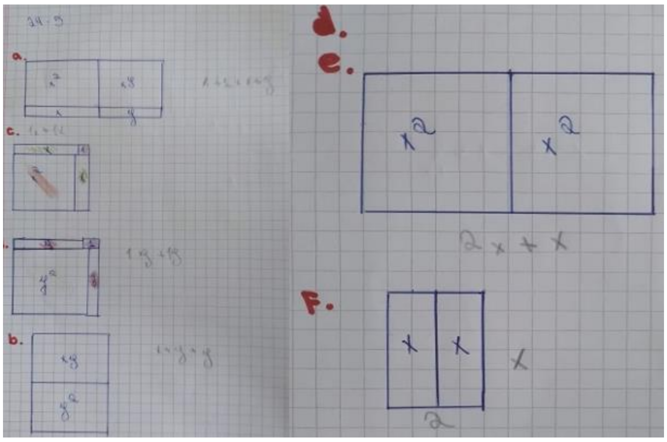


Figure 14. Example-4 (reprinted with permission of the student)

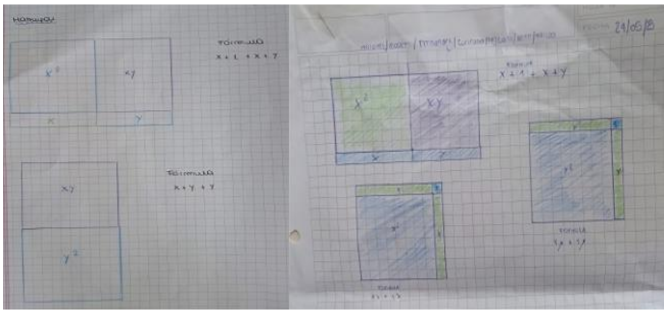


Figure 15. Example-5 (reprinted with permission of the student)

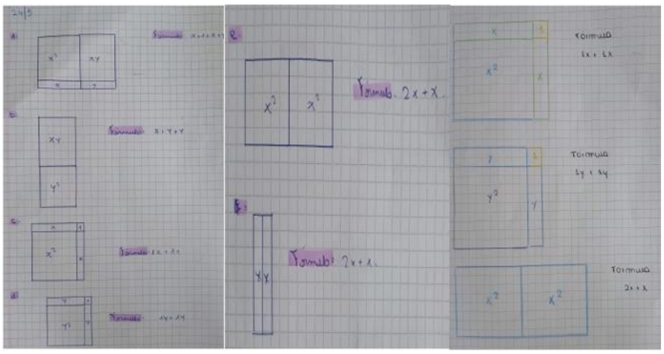


Figure 16. Example-6 (reprinted with permission of the student)

allows for processes of generalization associated with the areas of the figures (see **Figure 13**).

Among the findings of the students' work, it is noteworthy that they recognized issues related to commutativity in the representation of equivalent figures: side 1: x+1, side 2: x is equivalent to side 1: x, side 2: x+1. For the students, this representation signifies a rotation of the figure that determines the same assigned area value and the same dimensions (**Figure 14**, **Figure 15**, & **Figure 16**).

Class 4: During the development of this activity, students with a better understanding and confidence in the work carried out in part 1 of this class were able to recognize the lengths of the sides more clearly and

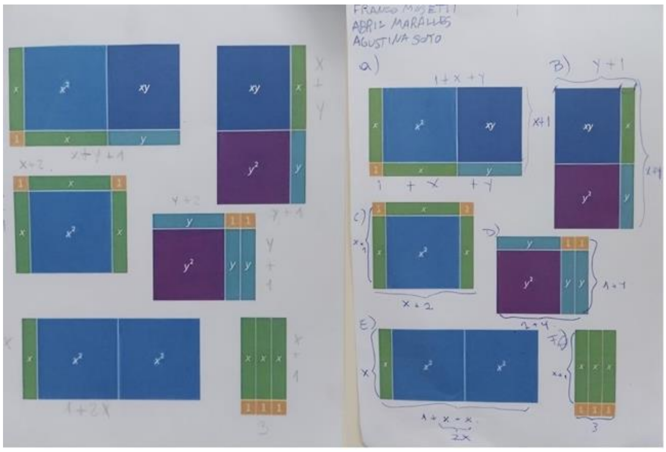


Figure 17. Example-7 (reprinted with permission of the student)

Figure 18. Example-8 (reprinted with permission of the student)

fluently. However, with regard to what the children called the "formula," they did not establish direct relationships between the concept of area as the product of the sides of a rectangular arrangement, assigning to the notion of area an additive idea of the measurement of the sides (**Figure 17** & **Figure 18**).

The activities scheduled for this session flowed more quickly. The students very skillfully determined the measurements and dimensions of the figures.

For the children, it was no longer a major challenge, as they already recognized the values assigned to the measurement of each tile.

During the motivation phase with virtual manipulatives in Mathigon, the students showed notable enthusiasm and motivation. They remembered and applied what they had learned previously, demonstrating skills in recognizing the lengths of the sides of the presented figures. Furthermore, their active participation and free play with Mathigon allowed them

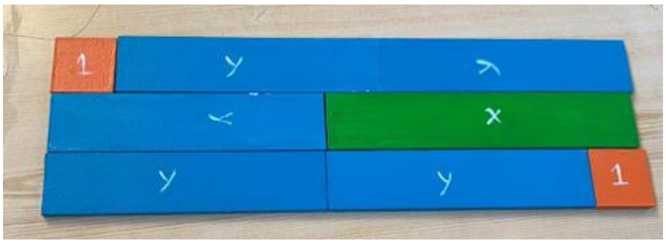


Figure 19. Example-9 (reprinted with permission of the student)

to creatively explore and find other equivalent figures. This fun and motivating experience in Mathigon boosted the students' interest and participation, contributing to a conducive learning environment and strengthening their intrinsic motivation for mathematics.

Class 5 & class 6: In this phase, the students did not encounter significant difficulties. They were able to effectively use Mathigon's tools to explore different figures and experiment with the lengths of their sides. Their active participation and ability to find other equivalent figures demonstrated their understanding and application of the previously worked concepts.

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It is evident that the focus on free exploration with Mathigon has been beneficial for the students. It has allowed them to develop their creativity, autonomy, and the ability to generalize geometric properties. By encouraging the autonomous application of their knowledge, they have discovered new relationships and patterns among geometric figures, demonstrating a deep understanding and advanced mathematical thinking.

Regarding the specific difficulties that some students faced when writing reducible algebraic expressions, it is understandable that each student has a different learning pace. The lack of a complete understanding of how the measurement of the algebratil "x" relates to the variable "x" in the algebraic expression is a common challenge in learning algebra. However, it is encouraging that clear explanations, additional examples, guided practice, and individualized feedback have been provided to address these difficulties.

The availability of explanatory videos on the Mathigon platform has also been valuable in clarifying doubts and strengthening students' understanding. This combination of resources and pedagogical strategies has contributed to supporting the students' learning process and facilitating their progress in writing reducible algebraic expressions.

In general, an environment of learning has been created in which students feel motivated and supported in their process of understanding and mastering algebraic concepts. This student-centered approach and the use of interactive tools like Mathigon are fundamental in promoting effective and lasting learning in mathematics. Continuing to provide support and practice opportunities will be key to the ongoing success of the students in this field.

It is essential to keep in mind that each student has their own learning pace, so constant practice and constructive feedback are essential for their progress. In this sense, the goal is to provide a learning environment where students feel motivated and supported in their process of understanding and mastering algebraic concepts.

At the end of the session, the teacher summarized the concepts and strategies that the students had learned through synthesis strategies in the classroom. She emphasized the importance of understanding the relationship between algebratils and reducible algebraic expressions, reminding them that each algebratil represents a variable or numerical coefficient in the expression. Through examples and manipulation of algebratils, the teacher reinforced the idea that each algebratil represents a variable or numerical coefficient in the algebraic expression. As a result, the students consolidated their understanding of the concepts and strategies covered during the session.

The teacher also provided feedback on the concept of distributive property and how it applies when multiplying a coefficient by each term within the parentheses, which simplifies and reduces polynomials efficiently. She reviewed the quadratic formula and problem-solving strategies. The idea that continuous practice is fundamental for improvement in mathematics was reinforced, and the students were encouraged to continue exploring and using the Mathigon Polypad tool to strengthen their understanding and skills in writing reducible algebraic expressions.

Furthermore, it was emphasized that facing challenges and difficulties in the learning process is normal, and students were encouraged not to give up and to seek support and guidance when needed.

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The motivation phase in the described activity lies in presenting polynomial examples to the students. By presenting these examples, the goal is to spark their interest and curiosity in the topic of polynomials and create a motivating learning environment. Showing various polynomial examples and challenging the students to represent them using the manipulative and their general and reducible algebraic expressions provides them with an opportunity to actively engage in the activity and explore the world of polynomials in a practical and tangible way through the Mathigon platform.

During this phase, the students had the opportunity to explore and understand polynomials more deeply. By using the manipulative, they could visualize how polynomials are formed and how the terms interact with each other. This allowed them to better grasp the structure and composition of polynomials. Additionally, by expressing polynomials in their general and reducible algebraic forms, the students applied mathematical rules and principles to simplify and combine similar terms. This helped them understand the importance of numerical coefficients, exponents, and terms in the algebraic representation of polynomials.

Throughout this phase, the students also established connections between the visual representation of the manipulative and the algebraic representation of polynomials. This allowed them to see how algebraic expressions relate to objects and how they can be used to solve mathematical problems.

During the activity, the teacher made observations and provided feedback to the students regarding their understanding of polynomials and their ability to represent them using both the Mathigon manipulative and their algebraic expressions. This continuous feedback allowed the students to correct errors and improve their performance as they progressed through the activity.

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In the closing phase of the activity, some time was allocated to review the challenges encountered in previous activities. The goal was to hear the students' experiences and opinions about using Mathigon. It was truly rewarding to discover that the students found Mathigon to be a very useful tool for understanding algebraic expressions and visualizing them more clearly. During this phase, the students shared their perspectives and mentioned that they found the tool easy to use, which enabled them to better grasp mathematical concepts. It was exciting to see how they could explore and experiment with algebraic expressions interactively, helping them strengthen their understanding and develop their mathematical skills in a more dynamic way. The closing activity not only provided the students with an opportunity to express their experiences and emotions but also allowed them to reflect on their achievements and the challenges they had overcome. It was a chance to consolidate their learning and establish connections between mathematical concepts and their practical application in Mathigon.

CONCLUSIONS & DISCUSSION

Here are the conclusions of the work carried out, outlined in light of two categories: the possibilities of the platform and the challenges and issues regarding its use.

Possibilities of Using Mathigon

One notable aspect of using Mathigon is the creation of a collaborative learning environment, where students supported each other and shared their knowledge and experiences with Mathigon Polypad. Those students who felt more comfortable with the platform assisted their peers who faced difficulties, fostering collaboration, the exchange of ideas, and a deeper understanding of mathematical concepts.

To address this challenge, the teacher played a pivotal role by providing additional personalized feedback to the students. Detailed reviews of students' responses were conducted, identifying errors and offering clear and detailed explanations of misconceptions. Moreover, additional examples and complementary resources were provided to help students understand and correct their mistakes.

Personalized feedback allowed students to grasp concepts more profoundly and address areas, where improvement was needed. This, in turn, helped strengthen their understanding of algebra and enhance their ability to apply concepts effectively.

It's important to emphasize that while real-time automatic feedback was helpful for some students, personalized feedback provided by the teacher played a fundamental role in meeting individual students' needs and promoting more meaningful learning. The combination of platform-generated automatic feedback and personalized teacher feedback addressed challenges and provided appropriate support to students in their algebra learning journey.

The integration of Mathigon Polypad into the classroom yielded several significant benefits for both students and the teacher in the development of algebra learning. Various key advantages were experienced, enhancing the educational process.

Firstly, Mathigon Polypad offered students a highly interactive learning experience, allowing them to manipulate and experiment with mathematical objects in real-time. This interactivity facilitated the comprehension and exploration of abstract concepts tangibly. Furthermore, it promoted active learning by granting students greater participation in the classroom.

Additionally, the platform provided visual representations of algebraic concepts, making the understanding of abstract ideas more accessible. Students could observe how graphs changed as they manipulated objects on the screen, establishing visual connections between algebraic expressions and their graphical representations. This real-time visualization proved invaluable for reinforcing their understanding and mathematical reasoning.

Another highlighted benefit was that Mathigon Polypad allowed students to work at their own pace and explore mathematical concepts through activities and challenges tailored to their skill levels. This created a more flexible and personalized learning environment, where each student could address concepts individually, respecting their unique needs and learning pace.

The platform also encouraged collaboration and teamwork among students. Observations showed how they shared their explorations and solutions with their peers, promoting discussion, the exchange of ideas, and collaborative problem-solving. Mathigon Polypad fostered a collaborative learning environment, where students could support each other and construct knowledge collectively.

Furthermore, the wide range of resources offered by Mathigon Polypad, such as tutorials, examples, and practical activities, provided students with opportunities to delve deeper into concepts and practice their skills extensively. Additionally, the platform is regularly updated with new content and challenges, ensuring that students have access to a variety of up-to-date and relevant educational materials.

Therefore, Mathigon offers numerous possibilities due to its adaptability and accessibility. It proved to be a tool suitable for different learning styles and levels, allowing students with various backgrounds to benefit from the platform. It enabled them to approach concepts visually, manipulatively, and interactively. It also facilitated motivation and commitment by promoting the exploration and experimentation of mathematical concepts, maintaining students' interest, and encouraging active participation in activities. Additionally, its instant feedback capability allowed timely error correction and enhanced students' autonomy and self-regulation in learning. Mathigon prepared students for the digital world, familiarizing them with a relevant educational technological tool and providing practical skills in using technology for mathematics learning. Lastly, it promoted the transfer of skills, as students developed critical thinking and problem-solving skills that are applicable in various situations and disciplines. Thus, the application has endless possibilities for future studies.

Challenges & Issues in Using Mathigon

The use of Mathigon Polypad as a digital tool presented a series of challenges and issues that were effectively addressed. One of the initial challenges was the need for internet access and compatible devices, which proved to be a problem in environments with limited resources and poor connectivity, as was the case in this experience. To overcome this difficulty, a creative solution was implemented by creating wooden "algebratiles" that replicated the proportions and sizes of the platform. These algebra tiles allowed the participation of students who did not have access to technological resources and provided a hands-on and tangible experience, both for them and for those who did have access to Mathigon Polypad.

Another challenge was the learning and complete mastery of the platform. Although Mathigon Polypad offers an intuitive interface, students required additional time to become familiar with all its features and functionalities. Moreover, many of them had difficulties recalling prior algebraic content, which affected their ability to fully utilize the platform's resources and activities. In response, the teacher played a crucial role in providing additional support and guidance during the learning process. Review and reinforcement activities were conducted to recall the basic concepts of algebra, and personalized feedback, practical examples, and real-world contexts were provided to facilitate the understanding and application of mathematical concepts.

On the other hand, while Mathigon Polypad provides an interactive learning experience for mathematics, there was a recognized need to provide additional feedback to students to understand and apply mathematical concepts properly. The teacher was prepared to offer extra support and guidance, answering questions, resolving doubts, and providing clear and detailed explanations. Therefore, it was acknowledged that Mathigon Polypad was not the ideal option for all students. Some required more traditional teaching methods or had difficulties learning autonomously through the platform. Therefore, teaching was adapted to address different needs and learning styles. Traditional methods, such as classroom explanations, practical exercises on paper and pencil, and group activities, were implemented, complementing the use of Mathigon Polypad and providing different learning approaches.

In addition to the challenges mentioned above, the use of Mathigon Polypad for learning algebra posed other issues that required attention and adaptation from both the teacher and the students. One of these issues was related to the difficulty of understanding the relationship between abstract algebraic concepts and their visual representation in Mathigon Polypad. Despite the platform offering interactive and visual tools, some students faced difficulties in connecting algebraic symbols and equations with their graphical representation on the screen. To overcome this obstacle, it was necessary to provide additional explanations, concrete examples, and analogies that allowed students to understand how abstract mathematical concepts translate into visual representations within the platform.

In some cases, students encountered complex mathematical situations or problems in Mathigon Polypad that generated frustration or confusion. The platform offers a wide variety of challenges and activities, but some of them may exceed the students' current skill or knowledge level. In this situation, it was essential to provide additional support and guidance, breaking down the problems into smaller steps and providing step-by-step examples to help students effectively address and solve mathematical challenges.

Furthermore, some students experienced difficulties in transferring the concepts and skills acquired in Mathigon Polypad to other mathematical contexts and situations. They found it challenging to apply the learned concepts and solve mathematical problems outside the digital platform's environment. To address this issue, it was necessary to provide activities and exercises that encouraged knowledge transfer, such as contextualized problems and real-world situations that required the application of concepts acquired in Mathigon Polypad. Some students required more specific and detailed feedback to understand their errors and improve their understanding of algebra significantly. The automatic feedback in the platform generally relied on correct or incorrect answers, without addressing the specific errors made by each student. This posed an additional challenge, as some students needed a more detailed explanation of their mistakes and specific guidance on how to correct them.

In summary, we can affirm that the use of Mathigon Polypad presented challenges and issues that were effectively addressed. Creative solutions were found to overcome technological limitations, additional support was provided for learning and mastering the platform, collaboration among students was encouraged, and teaching was adapted to different needs. Through these strategies, an inclusive, enriching, and effective learning experience was ensured for all the students involved.

Author contributions: PABÁ: conceptualization, methodology, supervision, writing – original draft; **MC-C:** formal analysis, data curation, validation, writing – review & editing; **RSB:** conceptualization, validation, resources; **JJ-G:** supervision, visualization, writing – review & editing. All authors have sufficiently contributed to the study and agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Ethical statement: The authors stated that the study did not require an ethics committee approval. Permission to carry out the study was granted from the educational institution under study. The students were assured of anonymity and that they were allowed to withdraw if they were not comfortable. The authors further stated that the family parents signed the informed consent form, which was confirmation that participation in this study was voluntary. To guarantee anonymity and protect the respondents' identities, pseudonyms names were used in this study.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

Armando, M., Régner, I., & Ochs, M. (2023). Toward a tool against stereotype threat in math: Children's perceptions of virtual role models. In *Proceedings of the International Conference on Multimodal Interaction* (pp. 306-310). ACM. https://doi.org/10.1145/3610661.3616179

Babo, L., Pinto, C. M. A., Mendonça, J., Rasteiro, D. M. L. D., Caridade, C. M. R., Lavicza, Z., Ulbrich, E., Rahmadi, I. F., Anđić, B., Abrori, F. M., Kocadere, S. A., Petridis, K., Nikolopoulos, C. D., Kokkinos, E. A., & Vardiambasis, I. O. (2023). MATHematics DIGital escape rooms–Empowering students. In A. Mesquita, A. Abreu, J. V. Carvalho, & C. H. P. de Mello (Eds.), *Perspectives and trends in education and technology. Smart innovation, systems and technologies* (pp. 375-385). Springer. https://doi.org/10.1007/978-981-19-6585-2_34

Brousseau, G. (1986). Fondements et méthodes de la didactiques des mathématiques [Foundations and methods of mathematics teaching]. R*echerches en Didactique des Mathématiques, 7(2),* 4-10.

Cabero, J., & Marín, V. (2014). Posibilidades educativas de las redes sociales y el trabajo en grupo. Percepciones de los alumnos universitarios [Educational possibilities of social networks and group work. University students' perceptions]. *Comunicar* [*Communicate*], *42*.

Caro, M. (2015). *Los beneficios del aprendizaje adaptativo* [*The benefits of adaptive learning*]. http://ideasqueinspiran.com/2015/12/14/los-beneficios-del-aprendizaje-adaptativo/

Chacón-Castro, M., Aimacaña-Espinosa, L., & Jadán-Guerrero, J. (2021). Escape rooms: Mathematical challenges available to educators. In A. Mesquita, A. Abreu, J. V. Carvalho, & C. H. P. de Mello (Eds.), *Perspectives and trends in education and technology. Smart innovation, systems and technologies* (pp. 195-205). Springer. https://doi.org/10.1007/978-981-16-5063-5_16

Chiappe, A. (2016). Tendencias sobre contenidos educativos digitales en América Latina [Trends in digital educational content in Latin America]. *UNESCO*. http://unesdoc.unesco.org/images/0024/002456/245673s.pdf

da Silva, R., & Duarte, V. (2019). Didactic situations: The application of oval tangram in the education of angles in the 6th year of fundamental education. *International Journal of Emerging Technologies in Learning, 2*(1), 32-46. https://doi.org/10.31692/2595-2498.V2I1.71

Ferreira, J. F., & Mendes, A. (2020). Open and interactive learning resources for algorithmic problem solving. In *Formal Methods. FM 2019 International Workshops* (pp. 200-208). Springer. https://doi.org/10.1007/978-3-030-54997-8_13

García, M. D., Prieto, F. J. G., & Hurtado, I. G. (2017). Moodle y Facebook como herramientas virtuales didácticas de mediación de aprendizajes: Opinión de profesores y alumnos universitarios [Moodle and Facebook as virtual didactic tools for learning mediation: Opinion of university professors and students]. *Revista Complutense de Educación* [*Complutense Education Magazine*], *29*(3), 807-827. https://doi.org/10.5209/rced.53968

Guizado, J. V., & Ortiz, J. Á. R. (2022). Retos digitales del profesorado en gestión de la enseñanza virtual de matemáticas [Digital challenges for teachers in managing virtual mathematics teaching]. *Revista Venezolana de Gerencia* [*Venezuelan Management Magazine*], *27*, 390-408. https://doi.org/10.52080/rvgluz.27.7.26

Hodovaniuk, T., Makhometa, T., Tiahai, I., Medivedieva, M., & Pryshchepa, S. (2020). The use of ICT in the flip teaching of future mathematics teachers. In *Proceedings of the 16th International Conference on ICT in Education* (pp. 709-720). https://doi.org/10.31812/123456789/4457

Huda, M. E. (2019). The problematic: Teachers' pedagogical ability in using technology on mathematics learning of junior high school. *Journal of Physics, 1200*, 012009. https://doi.org/10.1088/1742-6596/1200/1/012009

Kechil, R., Mohd, A., & Mohammad, A. (2022). *Geogram: Digital game-based learning tool for improving basic geometric knowledge*. https://ir.uitm.edu.my/id/eprint/60342/1/60342.pdf

Keldgord, F., & Ching, Y.-H. (2022). Teachers' experiences with and perceptions of virtual manipulatives following the COVID-19 pandemic. *TechTrends, 66*, 957-967. https://doi.org/10.1007/s11528-022-00796-9

Kliziené, I., Taujanskiené, G., Augustiniene, A., Simonaitiené, B., & Cibulskas, G. (2021). The impact of the virtual learning platform EDUKA on the academic performance of primary school children. *Sustainability, 13*(4), 2268. https://doi.org/10.3390/su13042268

Lee, J. Y., Pyon, C. U., & Woo, J. (2023a). Digital twin for math education: A study on the utilization of games and gamification for university mathematics education. *Electronics, 12*(15), 3207. https://doi.org/10.3390/electronics12153207

López-García, A. (2023). Effectiveness of a teaching methodology based on the theory of historical thinking through active methods and digital resources in Spanish adolescents. *Frontiers in Education, 8*. https://doi.org/10.3389/feduc.2023.1175123

Martín Díaz, A., Alario-Hoyos, C., Estévez-Ayres, I., Delgado Kloos, C., & Fernández-Panadero, C. (2023). ABENEARIO: A system for learning early maths with ABN. *Education and Information Technologies, 28*, 12859-12881. https://doi.org/10.1007/s10639-023-11692-x

Martin, C. S., Harbour, K., & Polly, D. (2022). Examining how emergency remote teaching influenced mathematics teaching. *TechTrends, 66*(2), 338-350. https://doi.org/10.1007/s11528-022-00711-2

Martín-Gutiérrez, J., Mora, C. E., Añorbe-Díaz, B., & González-Marrero, A. (2017). Virtual technologies trends in education. *EURASIA Journal of Mathematics, Science and Technology Education, 13*(2), 469-486. https://doi.org/10.12973/eurasia.2017.00626a

Mato-Vázquez. D., Castro-Rodríguez M., & Pereiro-González, C. (2018). Análisis de materiales didácticos digitales para guiar y/o apoyar el proceso de enseñanza-aprendizaje de las matemáticas [Analysis of digital teaching materials to guide and/or support the teaching-learning process of mathematics]. *Revista d'Innovació Educativa* [*Educational Innovation Magazine*], *20*, 80-88. https://doi.org/10.7203/attic.20.12117

Mayasari, N., Hasanudin, C., & Fitrianingsih, A. (2020). The use of wingeom software in geometry subject, how is the learning outcomes of junior high school students? *Journal of Physics: Conference Series, 1477*, 042042. https://doi.org/10.1088/1742-6596/1477/4/042042

Moral-Sánchez, S. N., Sánchez-Compaña, M. ª. T., & Romero, I. (2022). Geometry with a STEM and gamification approach: A didactic experience in secondary education. *Mathematics, 10*(18), 3252. https://doi.org/10.3390/math10183252

Oliveira Jr, A. P., & Datori Barbosa, N. (2023). Tasks about impossible random events: A pedagogical game as a teaching tool for the early years of elementary school. *Acta Scientiae* [*Journal of Science*], *25*(6), 183-208. https://doi.org/10.17648/acta.scientiae.7654

Palma, G. A., Arteaga, M. J., Hermosilla, A. S., Delgado, A., & Lee, E. (2023). Development of a web system to improve and reinforce learning in mathematics in primary and secondary students in Peru. *International Journal on Recent and Innovation Trends in Computing and Communication, 11*(5), 23218169. https://doi.org/10.17762/ijritcc.v11i5s.6597

Polypad – virtual manipulatives – mathigon. (2023). https://mathigon.org/polypad

Rahayu, E. G. S., Juandi, D., & Jupri, A. (2021). Didactical design for distance concept in solid geometry to develop mathematical representation ability in vocational high school. *Journal of Physics: Conference Series, 1882*(1), 012077. https://doi.org/10.1088/1742-6596/1882/1/012077

Rumanová, L. (2022). Reflection of problem solving task and importance of a-priori analysis in preparing teacher for the teaching process. *TEM Journal, 11*(2), 837-841. https://doi.org/10.18421/tem112-42

Salas, R., Ramírez, J., Eslava, A., Castañeda, R., & de la Cruz, G. (2022). Teachers' perception of web games and mobile devices at the higher educational level during the COVID-19 pandemic. *Texto Livre* [*Free Text*], *15*, e37074. https://doi.org/10.35699/1983-3652.2022.37074

Sepeng, P. (2023). Unlocking mathematical potential–The impact of dynamic symmetry activities on learning and teaching mathematics. *Symmetry: Culture and Science, 34*(3), 297-316. https://doi.org/10.26830/symmetry_2023_3_297

Tetchueng, J.-L., Garlatti, S., & Laube, S. (2007). A didactic-based model of scenarios for designing an adaptive and context-aware learning system. In *Proceedings of the IEEE/WIC/ACM International Conference on Web Intelligence* (pp. 703-726). IEEE. https://doi.org/10.1109/wi.2007.4427180

Thompson, M., Leonard, G., Mikeska, J. N., Lottero, P. S., Maltese, A., Pereira, G., Hillaire, G., Waldron, R., Slama, R., & Reich, J. (2022). Eliciting learner knowledge: Enabling focused practice through an open-source online tool. *Behavioral Sciences, 12*(9), 324. https://doi.org/10.3390/bs12090324

Vergara, A. (2021) *¿Se aprende matemáticas en un modelo virtual?* [*Do you learn mathematics in a virtual model?*] https://observatorio.tec.mx/edu-bits-blog/aprendizaje-de-las-matematicas-en-un-modelo-virtual/

Zaikin, O., Malinowska, M., Bakhtadze, N., & Żyławski, A. (2017). Motivation and social aspects of competence-based learning process. *Procedia Computer Science, 112*, 1092-1101. https://doi.org/10.1016/j.procs.2017.08.131